Forecast combination and model averaging using predictive measures

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1 Introduction

- Combining forecasts robustifies and improves on individual forecasts (Bates & Granger (1969))

- Bayesian model averaging provides a theoretical motivation and performs well in practice (Min & Zellner (1993), Madigan & Raftery (1994), Jacobson & Karlsson (2004))

- BMA based on an in-sample measure of fit, the marginal likelihood

- We suggest the use of an out-of-sample, predictive measure of fit, the predictive likelihood

2 Forecast combination using Bayesian model averaging

- $\mathfrak{M} = \{\mathcal{M}_1, \ldots, \mathcal{M}_M\}$ a set of possible models under consideration
  - Likelihood function $L(\mathbf{y}|\,\theta_i, \mathcal{M}_i)$
  - Prior probability for each model, $p(\mathcal{M}_i)$
  - Prior distribution of the parameters in each model, $p(\theta_i|\,\mathcal{M}_i)$
- Posterior model probabilities

$$p(\mathcal{M}_i|\,\mathbf{y}) = \frac{m(\mathbf{y}|\,\mathcal{M}_i)\,p(\mathcal{M}_i)}{\sum_{j=1}^{M} m(\mathbf{y}|\,\mathcal{M}_j)\,p(\mathcal{M}_j)}$$

$$m(\mathbf{y}|\,\mathcal{M}_i) = \int L(\mathbf{y}|\,\theta_i, \mathcal{M}_i)\,p(\theta_i|\,\mathcal{M}_i)\,d\theta_i$$

with $m(\mathbf{y}|\mathcal{M}_i)$ the *prior predictive density* or *marginal likelihood*

- Model averaged posterior

$$p(\phi|\,\mathbf{y}) = \sum_{j=1}^{M} p(\phi|\,\mathbf{y},\mathcal{M}_j)\,p(\mathcal{M}_j|\,\mathbf{y})$$

for ϕ some function of the parameters

  - Accounts for model uncertainty
  - In particular

$$\hat{y}_{T+h} = E(y_{T+h}|\,\mathbf{y}) = \sum_{j=1}^{M} E(y_{T+h}|\,\mathbf{y},\mathcal{M}_j)\,p(\mathcal{M}_j|\,\mathbf{y})$$

- Choice of models

  - Posterior model probabilities, $p(\mathcal{M}_i|\,\mathbf{y})$
  - Bayes factor

$$BF_{ij} = \frac{P(\mathcal{M}_i|\mathbf{y})}{P(\mathcal{M}_j|\mathbf{y})}\bigg/\frac{P(\mathcal{M}_i)}{P(\mathcal{M}_j)} = \frac{m(\mathbf{y}|\,\mathcal{M}_i)}{m(\mathbf{y}|\,\mathcal{M}_j)}$$

3 The predictive likelihood

- Split the sample $\mathbf{y} = (y_1, y_2, \ldots, y_T)'$ into two parts with m and l observations, with $T = m + l$.

$$\mathbf{y}_{T\times 1} = \begin{bmatrix} \mathbf{y}^*_{m\times 1} \\ \widetilde{\mathbf{y}}_{l\times 1} \end{bmatrix} \begin{matrix} \text{traning sample} \\ \text{hold-out sample} \end{matrix}$$

- The training sample $\mathbf{y}^*$ is used to convert the prior into a posterior

$$p\left(\theta_i | \mathbf{y}^*, \mathcal{M}_i\right)$$

- Leads to *posterior predictive density* or *predictive likelihood* for the hold-out sample $\widetilde{\mathbf{y}}$

$$p\left(\tilde{\mathbf{y}} \middle| \mathbf{y}^*, \mathcal{M}_i\right) = \int_{\theta_i} L\left(\tilde{\mathbf{y}} \middle| \theta_i, \mathbf{y}^*, \mathcal{M}_i\right) p\left(\theta_i \middle| \mathbf{y}^*, \mathcal{M}_i\right) d\theta_i$$

- *Partial* Bayes factors

$$PBF_{ij} = \frac{p\left(\tilde{\mathbf{y}} \middle| \mathbf{y}^*, \mathcal{M}_i\right)}{p\left(\tilde{\mathbf{y}} \middle| \mathbf{y}^*, \mathcal{M}_j\right)} = \frac{m\left(\mathbf{y} \middle| \mathcal{M}_i\right)}{m\left(\mathbf{y} \middle| \mathcal{M}_j\right)} \Bigg/ \frac{m\left(\mathbf{y}^* \middle| \mathcal{M}_i\right)}{m\left(\mathbf{y}^* \middle| \mathcal{M}_j\right)}$$

- Asymptotically consistent model choice requires $T/m \to \infty$
- *Predictive* weights for forecast combinations

$$p\left(\mathcal{M}_i\middle|\, \tilde{\mathbf{y}}, \mathbf{y}^*\right) = \frac{p\left(\tilde{\mathbf{y}}\middle|\, \mathbf{y}^*, \mathcal{M}_i\right) p\left(\mathcal{M}_i\right)}{\sum_{j=1}^{M} p\left(\tilde{\mathbf{y}}\middle|\, \mathbf{y}^*, \mathcal{M}_j\right) p\left(\mathcal{M}_j\right)}$$

- Can use improper priors on parameters of the models
- Forecast combination is based on weights from predictive likelihood
- Model specific posteriors based on the full sample
- Additional complication: How to choose the size of the training sample, m, and the hold-out sample, l?

3.1 Small sample results

- Linear model

$$\mathbf{y} = \mathbf{Z}\gamma + \varepsilon$$
$$\mathbf{Z} = (\iota, \mathbf{X})$$

- Prior

$$\gamma \sim N\left(\mathbf{0}, c\sigma^2 \left(\mathbf{Z}'\mathbf{Z}\right)^{-1}\right)$$
$$p\left(\sigma^2\right) \propto 1/\sigma^2$$

- The predictive likelihood is given by

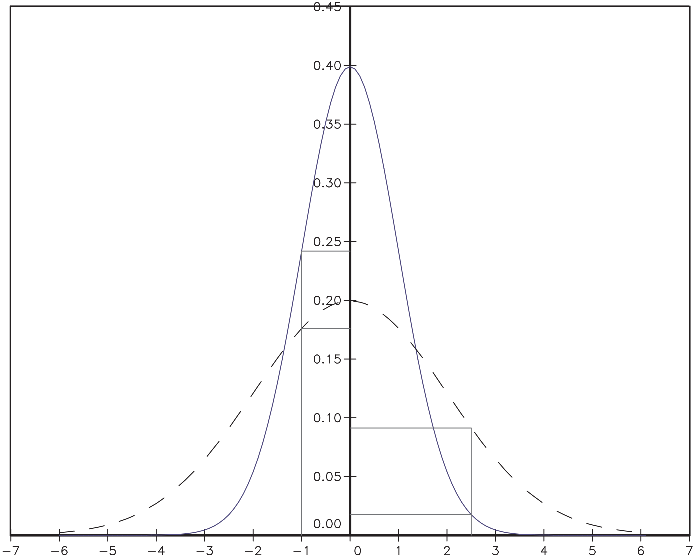
$$p\left(\tilde{\mathbf{y}}\right) \propto \left(\frac{S^*}{m}\right)^{-l/2} \frac{\left|\mathbf{M}^*\right|^{\frac{1}{2}}}{\left|\mathbf{M}^* + \widetilde{\mathbf{Z}}'\widetilde{\mathbf{Z}}\right|^{\frac{1}{2}}}$$
$$\times \left[m + \frac{1}{(S^*/m)}\left(\widetilde{\mathbf{y}} - \widetilde{\mathbf{Z}}\gamma_1\right)'\left(\mathbf{I} + \widetilde{\mathbf{Z}}\left(\mathbf{M}^*\right)^{-1}\widetilde{\mathbf{Z}}'\right)^{-1}\left(\widetilde{\mathbf{y}} - \widetilde{\mathbf{Z}}\gamma_1\right)\right]^{-T/2}$$

$$
\begin{aligned}
S^* &= \frac{c}{c+1}\left(\mathbf{y}^* - \mathbf{Z}^*\widehat{\gamma}^*\right)'\left(\mathbf{y}^* - \mathbf{Z}^*\widehat{\gamma}^*\right) + \frac{1}{c+1}\mathbf{y}^{*\prime}\mathbf{y}^* \\
\gamma_1 &= \frac{c}{c+1}\widehat{\gamma}^*, \\
\mathbf{M}^* &= \frac{c+1}{c}\mathbf{Z}^{*\prime}\mathbf{Z}^*
\end{aligned}
$$

- Three components
  - In sample fit, $\left(S^*/m\right)^{-l/2}$
  - Dimension of the model, $\left|\mathbf{M}^*\right|^{\frac{1}{2}} \Big/ \left|\mathbf{M}^* + \widetilde{\mathbf{Z}}'\widetilde{\mathbf{Z}}\right|^{\frac{1}{2}}$
  - Out of sample prediction,

$$
\left[m + \frac{1}{\left(S^*/m\right)}\left(\widetilde{\mathbf{y}} - \widetilde{\mathbf{Z}}\gamma^*\right)'\left(\mathbf{I} + \widetilde{\mathbf{Z}}\left(\mathbf{M}^*\right)^{-1}\widetilde{\mathbf{Z}}'\right)^{-1}\left(\widetilde{\mathbf{y}} - \widetilde{\mathbf{Z}}\gamma^*\right)\right]^{-T/2}
$$

Figure 1 Predictive likelihood for models with small and large prediction error variance.



4 MCMC

- Impossible to include all models in the calculations
  - Reduce the number of models by restricting the maximum number of variables to k'
  - Only consider "good" models
- Use reversible jump MCMC to identify good models
- Exact posterior probabilities calculated conditional on the set of visited models

Algorithm 1 Reversible jump Markov chain Monte Carlo

Suppose that the Markov chains is at model $\mathcal{M}$, having parameters $\theta_{\mathcal{M}}$.

1. Propose a jump from model $\mathcal{M}$ to a new model $\mathcal{M}'$ with probability $j(\mathcal{M}'|\mathcal{M})$.
2. Accept the proposed model with probability

$$\alpha = \min\left\{1, \frac{p\left(\widetilde{\mathbf{y}}|\mathbf{y}, \mathcal{M}'\right) p\left(\mathcal{M}'\right) j\left(\mathcal{M}|\mathcal{M}'\right)}{p\left(\widetilde{\mathbf{y}}|\mathbf{y}, \mathcal{M}\right) p\left(\mathcal{M}\right) j\left(\mathcal{M}'|\mathcal{M}\right)}\right\}$$

3. Set $\mathcal{M} = \mathcal{M}'$ if the move is accepted otherwise remain at the current model.

- Two types of moves

    1. Draw a variable at random and drop it if it is in the model or add it to the model (if $k_{\mathcal{M}} < k'$). This step is attempted with probability p_A.
    2. Swap a randomly selected variable in the model for a randomly selected variable outside the model (if $k_{\mathcal{M}} > 0$). This step is attempted with probability $1 - p_A$.

5 Simulation results

- Investigate the effect of the size of the hold-out sample
- Same design as Fernández, Ley & Steel (2001).
  - 15 possible predictors, $\mathbf{x}_1, \ldots, \mathbf{x}_{10}$ generated as $NID\,(0,1)$ and

$$(\mathbf{x}_{11}, \ldots, \mathbf{x}_{15}) = (\mathbf{x}_1, \ldots, \mathbf{x}_5)\,(0.3, 0.5, 0.7, 0.9, 1.1)'\,(1, \ldots, 1) + \mathbf{e},$$

    where $\mathbf{e}$ are $NID\,(0,1)$ errors.
  - Dependent variable

$$y_t = 4 + 2x_{1,t} - x_{5,t} + 1.5x_{7,t} + x_{11,t} + 0.5x_{13,t} + \varepsilon_t,$$

    with $\varepsilon_t \sim N\,(0, 6.25)$

- $\mathfrak{M}-$closed view, true model assumed to be part of the model set
- $\mathfrak{M}-$open view, variables x_1 and x_7 excluded from set of possible predictors
- Three data sets, with last 20 observations set aside for forecast evaluation
  - $T = 120$ (30 years of quarterly data),
  - $T = 250$
  - $T = 400$
  - 100 samples of each sample size
- Prior on models
$$p\left(M_j\right) \propto \delta^{k_j}\left(1-\delta\right)^{k'-k_j},$$
where k_j is the number of variables included in model j, $k' = 15$ and $\delta = 0.2$.
- g-prior with $c = k'^3 = 3375$

- The Markov chain is run for 70 000 replicates, with the first 20 000 draws as burn-in
- Suggests that 70-80% of the data should be kept for the hold-out sample

Table 1 RMSFE for simulated data sets

	min for l	PL	ML
small data set,$\mathfrak{M}$-closed	83	2.6333	2.6406
medium data set,$\mathfrak{M}$-closed	177	2.5064	2.5268
medium data set,$\mathfrak{M}$-open	182	3.5919	3.6499
large data set,$\mathfrak{M}$-closed	322	2.5308	2.5310
large data set,$\mathfrak{M}$-open	302	3.3956	3.4605

$\mathfrak{M}$-closed model :

$$y_t = 4 + 2x_{1,t} - x_{5,t} + 1.5x_{7,t} + x_{11,t} + 0.5x_{13,t} + \sigma\varepsilon_t, \tag{1}$$

with standard deviation 2.5

$\mathfrak{M}$-open model:

$$\begin{aligned} y_t|\, x_{-1,-7} = &-1.034x_{2,t} - 1.448x_{3,t} - 1.862x_{4,t} - 3.276x_{5,t} + 1.414x_{11,t} \\ &+ 0.414x_{12,t} + 0.914x_{13,t} + 0.414x_{14,t} + 0.414x_{15,t} \end{aligned} \tag{2}$$

with standard deviation 3.355.

Figure 2 Ratio of RMSFE for predictive likelihood and marginal likelihood as a function of l for the simulated medium data set.

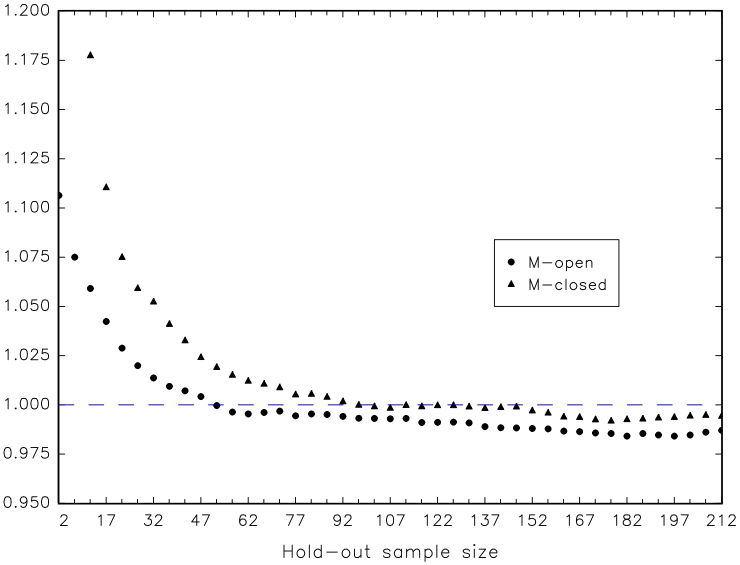


Figure 3 Ratio of RMSFE for predictive likelihood and marginal likelihood as a function of l for the simulated large data set.

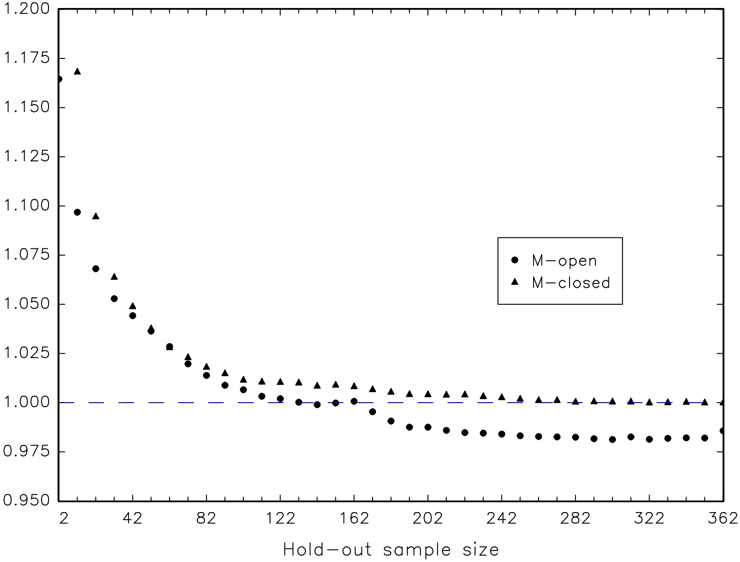


Figure 4 Variable inclusion probabilities (average) for large data set, $\mathfrak{M}$−closed view

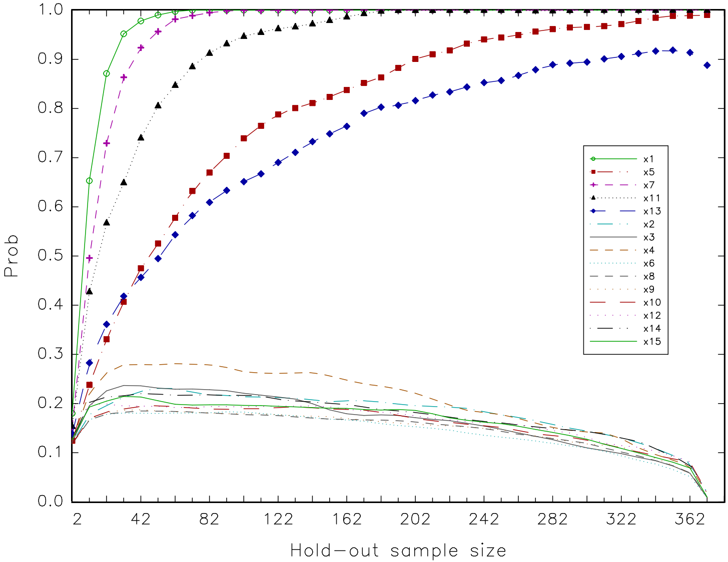
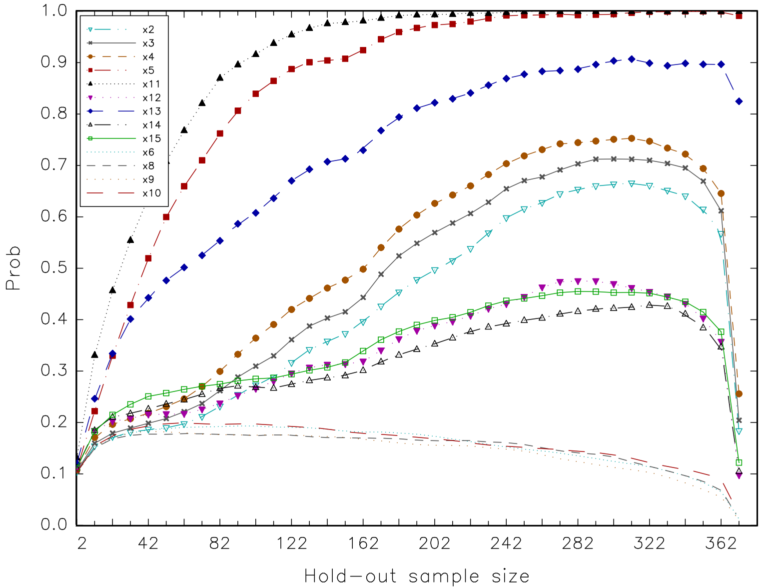


Figure 5 Variable inclusion probabilities (average) for large data set, $\mathfrak{M}$–open view



6 Swedish inflation

- Simple regression model of the form

$$y_{t+h} = \alpha + \omega d_{t+h} + \mathbf{x}_t \beta + \varepsilon_t,$$

- Constant term and a dummy variable, d_t, for the low inflation regime starting in 1992Q1 always included
- Quarterly data for the period 1983Q1 to 2003Q4 on 77 predictor variables
- Dynamics
  - A preliminary run is used to select, $\mathbf{x}_t^*$, the 20 most promising predictors
  - The final run is based on these with one additional lag

$$y_{t+h} = \alpha + \omega d_{t+h} + \mathbf{x}_t^* \beta_1 + \mathbf{x}_{t-1}^* \beta_2 + \varepsilon_t,$$

- 4 quarter ahead forecasts for the period 1999Q1 to 2003Q4

- Maximum of 15 predictors, $(k' = 15)$, $\delta = 0.1$
- $T = 64$, $l = 44$ for the hold-out sample
- 5 000 000 replicates

Table 2 RMSFE of the Swedish inflation 4 quarters ahead forecast, for $l = 44$.

	Predictive likelihood	*Marginal likelihood*
Forecast combination	0.9429	1.5177
Top 1	1.0323	1.5376
Top 2	0.9036	1.7574
Top 3	0.9523	1.6438
Top 4	1.0336	1.4828
Top 5	0.9870	2.0382
Top 6	0.9661	1.6441
Top 7	1.0534	1.5755
Top 8	1.1758	1.2905
Top 9	1.0983	1.8356
Top 10	1.0999	1.7202
Random walk	1.0251	1.0251

Figure 6 Swedish data, 4 quarters ahead inflation forecast, $l = 44$.

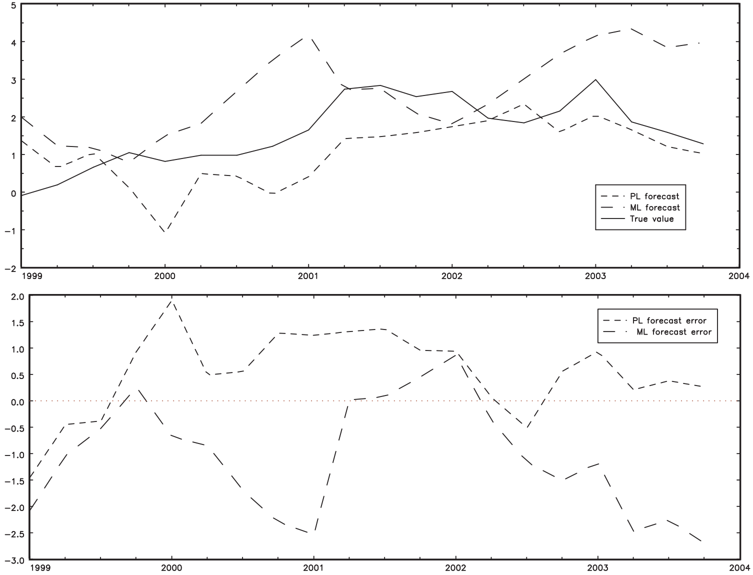


Table 3 Variables with highest posterior inclusion probabilities (average).

	Predictive likelihood		*Marginal likelihood*	
	Variable	Post. prob.	Variable	Post. prob.
1.	`Infla`	0.5528	`Pp1664`	0.9994
2.	`InfRel`	0.4493	`Pp1529`	0.9896
3.	`U314W`	0.3271	`InfHWg`	0.9456
4.	`REPO`	0.2871	`AFGX`	0.8104
5.	`IndProd`	0.2459	`PpTot`	0.4996
6.	`ExpInf`	0.2392	`PrvEmp`	0.4804
7.	`R5Y`	0.1947	`InfCns`	0.4513
8.	`InfFl`	0.1749	`InfPrd`	0.4105
9.	`M0`	0.1533	`R3M`	0.4048
10.	`InfUnd`	0.1473	`Pp75+`	0.3927
11.	`LabFrc`	0.1409	`ExpInf`	0.3829
12.	`NewHouse`	0.1245	`InfFor`	0.3786
13.	`InfImpP`	0.1225	`M0`	0.1793
14.	`PrvEmp`	0.1219	`POilSEK`	0.1702
15.	`PPP`	0.1134	`USD`	0.1170

Table 4 Posterior model probabilities, 4 quarters ahead forecast for 1999Q1 using predictive likelihood with $l = 44$.

	Model				
Variable	1	2	3	4	5
`InfRel`	×	×	×		×
$\texttt{InfRel}_{-1}$				×	×
`ExpInf`	×	×	×	×	×
`R5Y`	×	×	×		×
`InfFl`	×	×		×	×
$\texttt{InfFl}_{-1}$			×		
`InfUnd`	×	×	×	×	×
`USD`	×	×	×		×
`GDPTCW`		×			×
$\texttt{GDPTCW}_{-1}$				×	
Post. Prob	0.0538	0.0301	0.0218	0.0187	0.0184

Table 5 Posterior model probabilities, 4 quarters ahead forecast for 1999Q1 using marginal likelihood.

	Model				
Variable	1	2	3	4	5
Pp1664	×	×	×	×	×
Pp1529	×	×	×	×	×
InfHWg	×	×	×	×	×
$\texttt{AFGX}_{-1}$	×	×	×		
PpTot	×	×	×		×
$\texttt{PpTot}_{-1}$				×	
$\texttt{R3M}_{-1}$	×	×	×	×	×
InfFor	×				
$\texttt{InfFor}_{-1}$		×			
POilSEK			×		
$\texttt{NewJob}_{-1}$	×	×			
PP2534	×	×			
Post. Prob	0.1316	0.0405	0.0347	0.0264	0.0259

7 Conclusions

- The Bayesian approach to forecast combination works well
- The predictive likelihood improves on standard Bayesian model averaging based on the marginal likelihood
- The forecast weights based on predictive likelihood have good large and small sample properties
- Significant improvement when the true model or DGP not included in the set of considered models

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