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DYNIMO - Version III
A DSGE model of the Icelandic economy

By

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A DSGE model of the Icelandic economy

Stefán Thórarinsson*

December 3, 2020

Abstract

This handbook describes the third version of DYNIMO, the Central Bank of Iceland's dynamic stochastic general equilibrium model. Derivations from first principles to final equations are presented. In addition, the more advanced mathematical tools needed for the derivations are stated and their validity motivated. Subsequently, we estimate the model on Icelandic data over the period 2011Q1-2019Q4. Finally, evaluation of the model's properties is performed.

Keywords — DSGE, Iceland, macroeconomic model, monetary policy, Bayesian estimation, microfoundations.

JEL Classification: C32, C51, F41

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1 Introduction

This handbook documents version III of DYNIMO (**dyn**amic **I**celandic **mo**del), a small open-economy New-Keynesian DSGE model estimated on Icelandic data. DSGE models encompass a broad class of models, all of which are partly characterised by the properties described by each letter of the acronym:

- **D**ynamic models are characterised by the time dependence of agents' decisions.
- **S**tochastic fluctuations of states of nature drive deviations from the steady state.
- **G**eneral **E**quilibrium models, as opposed to partial equilibrium models, do away with assumptions of independence between sectors and markets. Furthermore, general equilibrium solutions of dynamic models contain optimal paths for all model variables at all periods considered.

It is illuminating to examine the evolution of models and economic thought leading to their conception. Earliest indications for what was to come can be found in the works of Frank Ramsey and Léon Walras. Walras pioneered the theory of general equilibrium, publishing *Éléments d'économie politique pure* (1874). Therein he lays the groundwork for general equilibrium theory, proving the existence of a solution and insisting that the price system has a coordinating function by aggregating agents' information into common knowledge of excess demand or excess supply. The primary contribution of Ramsey was the introduction of dynamics into general equilibrium theory in the 1928 paper *A mathematical theory of saving*, where he uses dynamic optimisation to maximise agents' utility with respect to savings. Although these contributions were highly regarded by their contemporaries, their profound effects on macroeconomic modelling were realised decades later. In the wake of more comprehensive and detailed national accounts, large-scale macroeconomic models came into vogue. Their golden age was the '50s and '60s with government bodies and central banks around the world employing such models for forecasting and analysis. These models grew out of the standard IS-LM models derived from Keynes' writings and consisted of ad hoc relationships and empirical correlations of macroeconomic variables. Disappointing performance of these models, in conjunction with the famous Lucas critique,[1] spurred the rational expectation revolution, spearheaded by Robert Lucas.[2] As a result, microfounded agent based models became more prevalent, leading to Kydland's and Prescott's seminal real business cycle (RBC) model. Their RBC model created fertile soil for the genesis of modern DSGE models. RBC style models presume a representative agent with time-independent preferences, characterised by deep parameters, operating in perfectly competitive good, asset, and factor markets. Uncertainty and dynamics are introduced through stochastic shocks to states of nature, to which the agent reacts in a manner consistent with rational expectations.[3] The representative agent is generally considered to be a Gorman aggregation of heterogeneous agents populating the economy.[4] Kydland's and Prescott's RBC model was calibrated to capture stylised facts of the US economy over the sample period. Extending the RBC model would have made parameter determination, be it calibration or estimation, a much more challenging task at that time. The development of a general solution to linear difference models

[1] An influential paper by Christopher Sims (1980) on VAR models aided in the downfall of large-scale macroeconomic models, but did so orthogonally to the rational expectations movement.

[2] The concept of rational expectations was first developed by John Muth in *Rational Expectations and the Theory of Price Movements* (Muth, 1961).

[3] In the early RBC models, the only source of uncertainty is technology.

[4] And thus necessarily implicitly assuming conditions that make such aggregation valid (see Gorman (1961) and related literature).

under rational expectation (see Blanchard and Kahn (1980)) and increased computing power opened up new avenues of research. The result was the modern DSGE model, which adopts the representative agent framework and the microfounded nature of the RBC models and builds on it. We can describe a canonical New-Keynesian DSGE model as a general equilibrium model derived from rational behaviour of a representative agent making time-dependent decision in reaction to stochastic fluctuations of states of nature, with added distortions. These distortions can take various forms. DYNIMO incorporates nominal rigidities and monopolistic competition, in addition to real rigidities in the form of habit persistence, variable capacity utilisation and sector specific adjustment-costs. Due to their microfoundation and added realism in form of distortions, New-Keynesian DSGE models provide a paradigm and theoretical framework for discussion and analysis of macroeconomic issues which is both plausible and rooted in rational behaviour.

The presented version of DYNIMO builds upon the original model developed by Seneca (2010) and a second version developed by Gestsson (2013) (henceforth referred to as DYNIMO I and DYNIMO II, respectively). DYNIMO I, in turn, is based on the foundational work of Woodford (2003), Christiano et al. (2005) and Smets and Wouters (2003), and uses NEMO and RAMSES, Norges bank's and Riksbank's DSGE models, respectively, as benchmarks.[5] The open economy characteristics of DYNIMO are inspired by Clarida et al. (2002). Seneca describes it hence: "The home economy is assumed to be a small open economy, which is modelled by letting its relative size go to zero in a general two-country model. International financial markets are incomplete and international financial intermediation is subject to endogenous costs as in Laxton and Pesenti (2003)." DYNIMO III is, by and large, a standard DSGE model, although it departs from the canonical DSGE model in a few ways to integrate peculiarities of the Icelandic economy. Most of the features of the Icelandic economy are relayed through the sign and intensity of dependence between variables, but on occasion the form of the relationships are altered to accommodate Icelandic features. Iceland is a small open economy which depends heavily on its export sector, a large part of which consists of a few but very different sub-sectors, namely tourism, aluminium and fisheries. The aluminium and fish industries do not respond to short-term price and demand fluctuations the same way as tourism and smaller export sectors do. Over the span of several years the aluminium industry is capacity constrained and the fishing industry is constrained by the decisions on total allowable catches, determined by the government on the basis of the state of the fish stock. In our setup, firms operating in these sectors are also considered to be price takers. In previous iterations of DYNIMO, the monetary policy rule was non-standard to further indulge Icelandic idiosyncrasies. The rationale was that since inflation targeting was only formally established in the year 2001, with a 2 year adaptation period, an appropriate monetary policy rule for the period 1991-2005 would not be restricted to responding solely to inflation and output as in the standard Taylor rule, but should include exchange rates as well. In the current iteration of the model, monetary authorities adhere more closely to the Taylor rule, responding to inflation, output gap and output growth.

Macroeconomic relationships derived from rational behaviour of agents should, in theory, be robust to structural shifts in the economy.[6] As a result DYNIMO is well suited for policy analysis. In particular, DYNIMO can disentangle the history of structural shocks, which sheds light on the mechanism driving the movement of aggregate macroeconomic variables. DYNIMO can further function as a forecasting model. Specifically, it can serve as a cross-check for other models with respect to comovements of dependent variables. DYNIMO is currently used for this

[5]See Brubakk et al. (2006) and Adolfsson et al. (2007) for the versions of the models Seneca cites. See Kravik and Mimir (2019) and Adolfsson et al. (2013) for later versions.

[6]With the caveat that all relevant structural shifts are represented in the model.

purpose at the Central Bank of Iceland (CBI), aiding the CBI's Quarterly Macroeconomic Model (QMM) (Daníelsson et al., 2019) in forecasting.

The handbook is structured as follows: In section 2 we describe the model's structure and underlying assumptions, as well as functional dependence of economic variables. In section 3 we specify the data, parameter calibration and the prior distributions of parameters to be estimated. We conclude the section by presenting the results of the estimation. In section 4 we report model properties. Specifically we compare the covariance matrices of the data and the model, produce impulse response functions and decompose the forecast error variance. In section 5 we derive the log-linearised version of the model. In section 6 we provide a summary of the linearised model. In Appendix A we detail derivations of all functional dependencies between variables. Appendix B contains detailed description of the data, the particulars of the data transformation, and specifications of the relationship between data and model variables. Additional steady state calculations are presented in Appendix C. Lastly, figures and tables are found in Appendix D.

2 Model description

2.1 General description

DYNIMO III is a general two-country model, where the world economy is populated by a continuum of households and firms. Domestic firms sell differentiated products to households at home and abroad in competition with foreign firms. Agents in the interval $[0, n]$, where $n \in [0, 1]$, make up the domestic economy, while agents in the interval $(n, 1]$, reside abroad. After specifying the model for an arbitrary n, we get the case of a small-open economy by letting n approach zero. Monopolistic competition is assumed and implemented within the Dixit-Stiglitz (1977) framework.

There are four broad types of agents in the model: households, firms, a central bank, and the government. Households lend capital and labour to firms, from which they receive rent and wages, respectively. Furthermore, households own the firms, from which they earn profits. In addition, households earn interest on bonds. Households use their earnings to finance consumption, acquire capital and government bonds, and to pay lump-sum taxes. Firms use labour and capital to produce goods and services. We presuppose two types of firms in the model to accommodate the capacity-constrained segment of the export sector. The central bank sets interest rates on bonds. The government levies lump-sum taxes to finance government consumption.

The agents interact in the labour, capital, bond, and goods markets. Households sell their labour to firms for a wage which is set by households as in Calvo (1983). Labour and capital are combined to produce intermediate goods. The intermediate goods are then priced by the firms within a Calvo pricing framework. The final good is made by the final good packer who costlessly combines intermediary goods into a final good which is used for consumption, investment, government consumption or maintenance. The capital market is assumed to be frictionless and perfectly competitive. One period risk-free state-contingent bonds are bought and sold by households, the price of which is directly influenced by the Central Bank.

2.1.1 Notation

An arbitrary variety absorbed domestically, $X_{k,t}(i,j)$, has the following arguments: k denotes the origin of the variety; i and j are the relevant agents which either decide on the magnitude of the variety or are affected by the magnitude; and t indicates the time period upon which these actions are taken. An aggregate of the variable $X_{k,t}(i,j)$ is denoted by $X_{k,t}(i)$, $X_{k,t}(j)$, or $X_{k,t}$, depending on the level of aggregation and the dimension along which the aggregation takes place. In addition, an aggregation of both domestic and foreign origins of the variable is denoted X_t. For example, consumption of a foreign variety i by a domestic household j at time t is denoted $C_{F,t}(i,j)$, and j's total consumption of imports is given by $C_{F,t}(j)$. The aggregate consumption of imports over all households is $C_{F,t}$. A step further would give an aggregation C_t of both $C_{H,t}$ and $C_{F,t}$. We differentiate between domestic and foreign absorption with a star superscript, e.g a domestic variety i, consumed by a foreign household j at time t is denoted by the quantity $C^*_{H,t}(i,j)$. The aggregation in the foreign absorption case is handled identically. In many cases we will solely focus on the domestic economy, omitting the corresponding specifications for the foreign economy.

For a given time dependent variable X_t, we will denote the corresponding stationary quantity by $\overline{X}_t$. If $X_t = \overline{X}_t$, the overline notation is generally omitted. Further, we write $x_t = \ln(\overline{X}_t)$, and denote the percentage deviation from the steady state, X, by $\widehat{X}_t = \frac{\overline{X}_t}{X} - 1 \approx x_t - x$. Lastly, expectations of a random variable X_t, conditional on information available at time τ is denoted $\mathbb{E}_\tau[X_t]$. The distinction between firm i and good i is minimal in our treatment and will be used almost interchangeably throughout the text.

2.2 Production

There are two types of firms producing intermediate goods in the domestic economy: The generic domestic firms producing differentiated goods for all markets with the same technology, and specialised export firms producing non-differentiated goods. The latter firms are characterised by perfect price inelasticity of supply, perfect substitutability, price taking and that their produce is solely absorbed abroad. In addition, we assume that they only employ capital in production. We use the subscript g for variables related to generic producers and E for variables relating to specialised exports.

2.2.1 Generic firms

Let $K_{g,t}(i)$ denote the capital a generic firm i has at its disposal and $N_{S,t}(i)$ the labour supply in hours that firm i employs in production. The generic firm produces a differentiated good, $Y_{g,t}(i)$, according to

$$Y_{g,t}(i) = K_{g,t}(i)^{\psi_H} (Z_t Z_{H,t} N_{S,t}(i))^{1-\psi_H} \tag{1}$$

where $0 \leq \psi_H \leq 1$, is the capital share parameter. The permanent economy-wide total factor productivity shock is given by Z_t. We denote its growth rate $\Pi_{Z,t} = \frac{Z_t}{Z_{t-1}}$, which we presume evolves according to

$$\Pi_{Z,t} = \Pi_{Z,t-1}^{\rho_Z} \Pi_Z^{1-\rho_Z} e^{\varepsilon_{Z,t}}, \quad \varepsilon_{Z,t} \sim N(0, \sigma_Z^2)$$

where $\rho_Z \in (0,1)$, $\Pi_Z > 0$, and $\sigma_Z > 0$ are constants. Note that the unconditional expectation of $\ln(\Pi_{Z,t})$ is $\ln(\Pi_Z)$, and we interpret Π_Z as the steady-state gross growth rate of technology.

The stationary shock process, $Z_{H,t}$, is a labour augmenting technology shock process, evolving in accordance with

$$Z_{H,t} = Z_{H,t-1}^{\rho_{Z_H}} e^{\varepsilon_{Z_H,t}}, \quad \varepsilon_{Z_H,t} \sim N(0, \sigma_{Z_H}^2), \quad \rho_{Z_H} \in (0,1)$$

We define the $N_{S,t} = N_t - \hbar N$, where N_t is the labour provided to firms by households and $\hbar N$ is interpreted as overhead labour.

Conditional factor demand in conjunction with cost minimisation yields (see Appendix A, section 7.3 for details)

$$MC_t(i) = MC_t = \frac{\psi_H^{-\psi_H}}{(1-\psi_H)^{\psi_H - 1}} \frac{W_t^{1-\psi_H} (R_t^K)^{\psi_H}}{(Z_t Z_{H,t})^{1-\psi_H}} \tag{2}$$

where W_t is the wage index and R_t^K is the nominal rental rate of capital. As expected, marginal cost is positively related to the price of factors of production, weighted by labour share, and inversely related to technology shocks. In fact, up to a scalar multiple, the marginal cost can be thought of as the weighted geometric mean of quality adjusted factor prices.

2.2.2 Specialised export firm

The specialised export firm produces a differentiated good, $Y_{E,t}(i)$, according to

$$Y_{E,t}(i) = Z_{E,t} K_{E,t}(i) \tag{3}$$

Since we assume that specialised export producers are capital constrained, there exists a constant, $\mathcal{K}$, such that $K_{E,t}(i) \leq \mathcal{K} Z_t$. The shock Z_t is the aforementioned permanent technology shock, and $Z_{E,t}$ is a stationary technology shock specific to specialised exports, given by

$$Z_{E,t} = Z_{E,t-1}^{\rho_E} e^{\varepsilon_{E,t}}, \quad \varepsilon_{E,t} \sim N(0, \sigma_E^2), \quad \rho_E \in (0,1)$$

2.3 Factor markets

2.3.1 Labour market

For a given firm, $i \in [0, n]$, we define its labour demanded, $N_t(i)$, via a labour packer as a Dixit-Stiglitz aggregate of the differentiated labour services supplied by domestic households:[7]

$$N_t(i) = n^{\frac{\rho_{W,t}-1}{\rho_{W,t}}} \left(\int_0^n N_t(i,j)^{\rho_{W,t}} dj \right)^{\frac{1}{\rho_{W,t}}} \tag{4}$$

where $\epsilon_{W,t} = \frac{1}{1-\rho_{W,t}} > 1$ is the stochastic elasticity of substitution between individual labour services.[8] Denoting the wage rate demanded by household j as $W_t(j)$, cost minimisation by firms for a given level of total labour input gives the following demand schedule for household j's labour services (see Appendix A, section 7.3, for details):

[7] We can omit the g subscript since only generic firms employ labour.

[8] The process will be defined in terms of markups in section 2.6.1, which implicitly defines the distribution of the elasticity of substitution.

$$N_t(j) = \left(\frac{W_t(j)}{W_t}\right)^{-\epsilon_{W,t}} N_t \tag{5}$$

where

$$W_t = \left(\frac{1}{n}\int_0^n W_t(j)^{1-\epsilon_{W,t}} dj\right)^{\frac{1}{1-\epsilon_{W,t}}} \tag{6}$$

is the wage index and

$$N_t = \frac{1}{n}\int_0^n N_t(i)di$$

is the total hours, per capita, demanded. Demand for a particular household's labour services depends negatively on the relative wages the household demands, and positively on the total labour service demanded in the economy.

2.3.2 Capital market

Households own the capital, $K_{S,t}(j)$, and rent $K_{E,t}(j)$ and $K_{g,t}(j)$ to the specialised export sector and generic firms, respectively, in a perfectly competitive rental market at the nominal rate R_t^K. Each household chooses the utilisation rate, $U_t(j)$, given by[9]

$$K_t(j) = U_t(j)K_{S,t}(j)$$

where $K_t(j) = K_{E,t}(j) + K_{g,t}(j)$. The cost of capital utilisation is given by

$$M_t(j) = \Gamma_U(U_t(j))K_{S,t}(j)$$

where Γ_U is an increasing convex function satisfying $\Gamma_U(1) = 0$. The capital accumulation equation is the standard:

$$K_{S,t+1} = (1-\delta)K_{S,t} + Z_{I,t}\left(1 - \Gamma_I\left(\frac{I_t}{I_{t-1}}\right)\right) I_t$$

The investment-specific technology shock, $Z_{I,t}$ determines the transformation efficiency of investment into capital, it evolves according to

$$Z_{I,t} = Z_{I,t-1}^{\rho_I} e^{\varepsilon_{I,t}}, \quad \varepsilon_{I,t} \sim N(0,\sigma_I^2), \quad \rho_I \in (0,1)$$

Investment adjustment cost is given by a non-negative convex function satisfying $\Gamma_I(\Pi_Z) = \Gamma_I'(\Pi_Z) = 0$ and $\Gamma_I''(\Pi_Z) > 0$. In words, investment adjustment cost is zero along the balanced growth path, but positive otherwise, and increasing the farther investment growth is from the balanced growth path. The aggregate effective capital per capita is given by

$$K_t = \frac{1}{n}\int_0^n K_t(j)dj$$

[9] We follow Christiano et al. (2005) in our specification of capital utilisation and its associated cost.

2.4 Demand aggregation

2.4.1 Domestic demand

Let the domestic households' consumption of domestic goods, $C_{H,t}(j)$, be dictated by a Dixit-Stiglitz aggregation:

$$C_{H,t}(j) = n^{\frac{\rho_{H,t}-1}{\rho_{H,t}}} \left(\int_0^n C_{H,t}^{\rho_{H,t}}(k,j)dk \right)^{\frac{1}{\rho_{H,t}}} \tag{7}$$

Domestic consumption of foreign goods is defined similarly:

$$C_{F,t}(j) = (1-n)^{\frac{\rho_{F,t}-1}{\rho_{F,t}}} \left(\int_n^1 C_{H,t}^{\rho_{F,t}}(k,j)dk \right)^{\frac{1}{\rho_{F,t}}} \tag{8}$$

where $\rho_{l,t}$, $l \in \{H, F\}$, is related to the elasticity of substitution, $\epsilon_{l,t}$, by $\rho_{l,t} = \frac{\epsilon_{l,t}-1}{\epsilon_{l,t}}$. Throughout we assume that $\epsilon_{l,t} > 1$. Total consumption, which enters domestic households j's utility function, is given by

$$C_t(j) = \left[\bar{\alpha}^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}}(j) + (1-\bar{\alpha})^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}}(j) \right]^{\frac{\eta}{\eta-1}} \tag{9}$$

where $1 - \bar{\alpha} = (1-n)\alpha$, and α is a proxy for the degree of openness. Conversely, $1 - \alpha$, is a measure of home bias.

As with consumption, let us define the price index of domestically produced goods within the Dixit-Stiglitz framework as

$$P_{H,t} = \left(\frac{1}{n} \int_0^n P_{H,t}^{1-\epsilon_{H,t}}(k)dk \right)^{\frac{1}{1-\epsilon_{H,t}}} \tag{10}$$

and similarly for imported goods we have

$$P_{F,t} = \left(\frac{1}{n-1} \int_n^1 P_{F,t}^{1-\epsilon_{F,t}}(k)dk \right)^{\frac{1}{1-\epsilon_{F,t}}} \tag{11}$$

Finally, we define the consumer price index, P_t, such that it satisfies:

$$P_t C_t = P_{H,t} C_{H,t} + P_{F,t} C_{F,t}$$

It can be shown that the consumer price index is equivalently written within the Dixit-Stiglitz framework as (see Appendix A, section 7.4, for details)

$$P_t = \left[\bar{\alpha} P_{H,t}^{1-\eta} + (1-\bar{\alpha}) P_{F,t}^{1-\eta} \right]^{\frac{1}{1-\eta}} \tag{12}$$

Optimising a domestic household's consumption decisions through expenditure minimisation for a given level of consumption yields the following aggregate demand schedules (see Appendix A, section 7.4, for details)

$$C_{H,t}(i) = \bar{\alpha} \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\epsilon_{H,t}} \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} C_t \tag{13}$$

for domestic goods and the demand for imported goods is given by

$$C_{F,t}(i) = \frac{n}{n-1}(1-\bar{\alpha})\left(\frac{P_{F,t}(i)}{P_{F,t}}\right)^{-\epsilon_{F,t}}\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t \tag{14}$$

where

$$C_t = \frac{1}{n}\int_0^n C_t(j)dj$$

is the total domestic consumption per capita. Similarly as for the labour services, the demand for a good i in either market depends negatively on the relative price of that good, while it increases with the total demand in the economy. The same relationship holds for foreign demand below.

2.4.2 Foreign demand

The foreign case differs from the domestic case due to the inclusion of an export adjustment cost and the existence of specialised export firms. For a foreign household j, we denote its demand for a generic domestic good as $C^*_{g,t}(j)$ and write $C^*_{E,t}(j)$ for the household's demand for specialised export goods. Let α_E be the bias towards the specialised export. The export adjustment cost, $\Gamma^*_{H,t}$, is a positive convex function and we define the effective foreign demand for domestic goods as

$$\check{C}^*_{H,t}(j) = [1-\Gamma^*_{H,t}]C^*_{H,t}(j) \tag{15}$$

where

$$C^*_{H,t}(j) = \left[(1-\alpha_E)^{\frac{1}{\eta_E}} C^*_{g,t}(j)^{\frac{\eta_E-1}{\eta_E}} + \alpha_E^{\frac{1}{\eta_E}} C^*_{E,t}(j)^{\frac{\eta_E-1}{\eta_E}}\right]^{\frac{\eta_E}{\eta_E-1}} \tag{16}$$

The final good which enters a foreign household's utility function is given by

$$C^*_t(j) = \left[(\bar{\alpha}^*)^{\frac{1}{\eta}} \check{C}^*_{H,t}(j)^{\frac{\eta-1}{\eta}} + (1-\bar{\alpha}^*)^{\frac{1}{\eta}} C^*_{F,t}(j)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \tag{17}$$

Other relevant quantities are defined analogously to the domestic case, with the exception of the price index, which is defined as

$$P^*_t C^*_t = \check{P}^*_{H,t}\check{C}^*_{H,t} + P^*_{F,t}C^*_{F,t} \tag{18}$$

where

$$\check{P}^*_{H,t} = P^*_{H,t}\left(\frac{\partial \check{C}^*_{H,t}}{C^*_{H,t}}\right)^{-1}$$

We assume further that the price of specialised exports is indexed to export prices and we thus have

$$P^*_{E,t} = P^*_{g,t} = P^*_{H,t}$$

The foreign demand schedule for domestic good i is given by (see Appendix A, section 7.4 for details):

$$C^*_{g,t}(i) = \left(\frac{1-n}{n}\right)\bar{\alpha}^*(1-\alpha_E)\left(\frac{P^*_{g,t}(i)}{P^*_{g,t}}\right)^{-\epsilon_{N,t}}\left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta} \cdot \Upsilon(\cdot) \tag{19}$$

$$C_{E,t}^*(i) = \left(\frac{1-n}{n}\right)\bar{\alpha}^*\alpha_E\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\eta}\cdot\Upsilon(\cdot) \tag{20}$$

where

$$\Upsilon_t = \frac{1}{1-n}\int_n^1 \left(\check{\Gamma}_{H,t}^*(j))^\eta \cdot C_t^*(j)\right) dj$$

and

$$\check{\Gamma}_{H,t}^*(j) = \frac{1-\Gamma_{H,t}^* - \frac{\partial\Gamma_{H,t}^*}{\partial C_{H,t}(j)}C_{H,t}^*(j)}{(1-\Gamma_{H,t}^*)^{\frac{1}{\eta^*}}}$$

The foreign demand schedule for a foreign good i is given by

$$C_{F,t}^*(i) = \left(\frac{n}{1-n}\right)\bar{\alpha}^*\left(\frac{P_{F,t}^*(i)}{P_{F,t}^*}\right)^{-\epsilon_{F,t}}\left(\frac{P_{F,t}^*}{P_t^*}\right)^{-\eta}\cdot C_t^* \tag{21}$$

with

$$C_t^* = \frac{1}{1-n}\int_n^1 C_t^*(j)dj$$

as the total foreign consumption per capita.

2.5 Household consumption and investment decisions

The discounted utility of an infinitely lived household j, at time τ, is given by

$$\mathbb{E}_\tau\left[\sum_{t=\tau}^{\infty}\beta^t\mathcal{U}_t(C_t(j), C_{t-1}, N_t(j), N_{t-1})\right]$$

where $\mathcal{U}_t = \left(Z_{C,t}\ln(C_t(j) - h_C C_{t-1}) - \chi\frac{(N_t(j)-h_N N_{t-1})^{1+\phi}}{1+\phi}\right)$, $\beta \in (0,1)$ is the subjective discount rate, and $Z_{C,t}$ is a preference shock, evolving according to

$$Z_{C,t} = Z_{C,t-1}^{\rho_C}e^{\varepsilon_{C,t}}, \quad \varepsilon_{C,t}\sim N(0,\sigma_C^2), \quad \rho_C \in (0,1)$$

The parameters $h_C, h_N \in (0,1)$, measure the degree of habit persistence in consumption and labour, respectively, and ϕ is the inverse Frisch elasticity of labour supply. Lastly, χ determines the equilibrium marginal rate of substitution.

The intertemporal budget constraint ensures that a domestic household's allocation of resources doesn't exceed available resources, and is given by

$$\begin{aligned} &P_t\left(C_t(j) + I_t(j) + M_t(j)\right) + \xi_t B_{H,t+1}^*(j) + \mathbb{E}_t[\Lambda_{t,t+1}B_{H,t+1}(j)] + TA_t(j) \\ =&R_{t-1}^*(1-\Gamma_{B,t-1})\xi_t B_{H,t}^*(j) + B_{H,t}(j) + W_t(j)N_t(j) + R_t^K U_t(j)K_{S,t}(j) + D_t(j) \end{aligned}$$

The left-hand side represents household j's allocation of resources to consumption, $C_t(j)$, investment, $I_t(j)$, and the cost of capital utilisation, $M_t(j)$. Further resources go towards acquiring foreign bonds, $\xi_t B_{H,t+1}^*(j)$, where ξ_t is the nominal exchange rate and is defined as the price of one unit of foreign currency measured in domestic currency. Another saving option for household

j is domestic contingent claims, the present value of which is given by $\mathbb{E}_t[\Lambda_{t,t+1} B_{H,t+1}(j)]$, where $\Lambda_{t,t+1}$ is the state-contingent stochastic discount rate.[10] Finally, a portion of the household's income goes to lump-sum taxes, $TA_t(j)$. The right-hand side gives available resources as the sum of holdings of foreign and domestic bonds, with foreign and domestic bond holdings acquired at time $t-1$ denoted by B_t^* and B_t, respectively; labour income, $W_t N_t$; rental income from capital $R_t^K U_T K_{S,t}$; and real dividends from firms, denoted by D_t. A foreign bond earns the foreign gross risk-free rate R_t^* and we define the risk-free gross interest rate by

$$R_t = \mathbb{E}[\Lambda_{t,t+1}]^{-1} \tag{22}$$

International financial friction is introduced to induce stationarity in net asset positions by specifying the transaction cost function:

$$\Gamma_{B,t-1} = \phi_1 \exp\left(\phi_2 \frac{\xi_{t-1} B_{H,t}^*}{P_{t-1} Y_{t-1}} - 1\right) + \ln(Z_{B,t-1}) \tag{23}$$

where $\phi_1 \in [0,1], \phi_2 > 0$ and

$$B_{H,t}^* = \frac{1}{n} \int_0^n B_{H,t}^*(j) dj$$

is the aggregate per capita holdings of foreign bonds by domestic households. With this specification, the country risk premium increases in the ratio of its foreign debt to GDP. The risk-premium shock to the transaction cost function is represented by $Z_{B,t}$ and evolves according to

$$Z_{B,t} = Z_{B,t-1}^{\rho_B} e^{\varepsilon_{B,t}}, \quad \varepsilon_{B,t} \sim N(0, \sigma_B^2), \quad \rho_B \in (0,1)$$

Solving the first order conditions, we get the following familiar optimality conditions[11] (See Appendix A, section 7.2, for details.)

$$1 = \mathbb{E}_\tau \left[-\frac{MU_{N,t}}{MU_{C,t}} \cdot \frac{P_t}{W_t} \right] \tag{24}$$

$$1 = R_t^* (1 - \Gamma_{B,t}) \mathbb{E}\left[\Lambda_{t,t+1} \frac{\xi_{t+1}}{\xi_t}\right] \tag{25}$$

$$\Gamma'_U(U_t) = \frac{R_t^K}{P_t} \tag{26}$$

$$Q_t = \mathbb{E}_t \left[\Lambda_{t,t+1} \frac{P_{t+1}}{P_t} \left(\frac{R_{t+1}^K}{P_{t+1}} U_{t+1} - \Gamma_{U,t+1}) + (1-\delta) Q_{t+1} \right) \right] \tag{27}$$

$$\begin{aligned} 1 =& Q_t Z_{I,t} \left(1 - \Gamma_{I,t}(\cdot) - \Gamma'_{I,t}(\cdot) \left(\frac{I_t}{I_{t-1}} \right) \right) \\ &+ \mathbb{E}_t \left[Q_{t+1} \frac{P_{t+1}}{P_t} \Lambda_{t,t+1} Z_{I,t+1} \Gamma'_{I,t+1}(\cdot) \left(\frac{I_{t+1}}{I_t} \right)^2 \right] \end{aligned} \tag{28}$$

[10] A state-contingent stochastic discount factor at time t is defined as the period-t price of a claim to one unit of consumption in a particular state in period $t+1$, divided by the period-t probability of that state occurring.

[11] We drop the reference to household j for clarity.

where

$$\Lambda_{t,t+1} = \beta \frac{MU_{C_{t+1}}}{MU_{C_t}} \frac{P_t}{P_{t+1}}$$

is the stochastic discount factor, Q_t is the marginal Tobin's q, and

$$MU_{C,t} = \frac{\partial \mathcal{U}_t(C_t(j), C_{t-1}, N_t)}{\partial C_t(j)} = Z_{C,t} \left(C_t(j) - hC_{t-1} \right)^{-1} \tag{29}$$

$$MU_{N,t} = \frac{\partial \mathcal{U}_t(C_t(j), C_{t-1}, N_t)}{\partial N_t(j)} = -\chi \left(N_t(j) - h_N N_{t-1} \right)^{\varphi} \tag{30}$$

are the marginal utilities of consumption and labour, respectively. Note that if there is no investment adjustment cost, i.e. $\Gamma_{I,t}(\cdot)$ is identically 0, then we get $Q_t = Z_{I,t}^{-1}$, as expected.

2.6 Prices, wages and exchange rates

2.6.1 Price setting

We employ the Calvo mechanism in modelling the price setting of firms, where a generic domestic firm i is allowed to set its domestic (export) price, at time $t+1$, with probability $1-\theta_H$ $(1-\theta_H^*)$ to a desired price $\mathcal{P}_{t+1}(i)$ $\left(\mathcal{P}_{t+1}^*(i)\right)$, while retaining a partially indexed price from the previous period with probability θ_H (θ_H^*). For domestic prices we thus define:

$$P_{H,t+1}(i) = \begin{cases} \mathcal{P}_{t+1}(i) & \text{with probability } (1-\theta_H) \\ \Pi_t^{\gamma_H} P_{H,t}(i) & \text{with probability } \theta_H \end{cases} \tag{31}$$

Similarly for export prices of the generic domestic firm we have:

$$P_{g,t+1}^*(i) = \begin{cases} \mathcal{P}_{t+1}^*(i) & \text{with probability } (1-\theta_H^*) \\ (\Pi_t^*)^{\gamma_H^*} P_{g,t}^*(i) & \text{with probability } \theta_H^* \end{cases}$$

For the specialised export firm we define

$$P_{E,t+1}^*(i) = P_{E,t+1}^* = P_{g,t}^* = \left(\frac{1}{n} \int_0^n (P_{g,t}^*(i))^{1-\epsilon_{H,t}^*} di \right)^{\frac{1}{1-\epsilon_{H,t}^*}}$$

For import prices we define price formation analogously to that of export prices of the generic domestic firm. Similarly, prices of foreign produced goods sold abroad evolve analogously to prices of domestically produced goods for the home market.

When a generic domestic firm i changes its price, it chooses a price which maximises the value of the firm, which in turn is determined by expected profits. We get the following maximisation problem (see Appendix A, section 7.6, for details):

$$\max_{\mathcal{P}_\tau, \mathcal{P}_\tau^*} \mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \Delta_{t|\tau,\tau} \left(\mathcal{A}_{t|\tau}(i) - \mathcal{C}\left(Y_{g,t|\tau}^d(i) \right) \right) \right] \tag{32}$$

where

$$\mathcal{A} = \theta_H^{t-\tau} A_{H,t|\tau}(i) P_{H,t|\tau}(i) + (\theta_H^*)^{t-\tau} \xi_{t|\tau} A_{H,t|\tau}^*(i) P_{H,t|\tau}^*(i)$$

denotes revenues, and where $\mathcal{C}$ is the cost function, $\Delta_{t,\tau}$ is the firm's discount rate,[12] and $A_{H,t}(i)$ and $A^*_{H,t}(i)$ are the domestic and foreign absorption of the domestic good i, respectively. Note that the subscript $t \mid \tau$ indicates the variables' value at time t conditional on that the last price decision was at time τ. The first order condition with respect to $\mathcal{P}_\tau$ is given by[13]

$$\begin{aligned} 0 = \mathbb{E}_\tau &\left[\sum_{t=\tau}^{\infty} \theta_H^{t-\tau} \Delta_{t|\tau,\tau} A_{H,t|\tau}(1-\epsilon_{H,t})\mathcal{P}_\tau \left(\frac{P_{t-1}}{P_{\tau-1}}\right)^{\gamma_H}\right] \\ &- \mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \theta_H^{t-\tau} \Delta_{t|\tau,\tau} A_{H,t|\tau}(1-\epsilon_{H,t})\mathcal{M}_{H,t} MC_{t|\tau}\right] \end{aligned} \tag{33}$$

where $\mathcal{M}_{H,t} = \frac{\epsilon_{H,t}}{\epsilon_{H,t}-1}$. Clearly elasticity of substitution is independent of price decisions and thus $\epsilon_{H,t|\tau} = \epsilon_{H,t}$. Furthermore, the general price level of the domestic economy is independent of which firms are allowed to optimise their prices and thus $P_{t|\tau} = P_t$ for all $t \in \mathbb{Z}$. Note that when domestic prices are perfectly flexible, i.e. $\theta_H = 0$, then the first order condition reduces to

$$\mathcal{P}_t = \mathcal{M}_{H,t} MC_t$$

and we get a natural interpretation of $\mathcal{M}_{H,t}$ as the domestic desired markup.[14] The elasticity of substitution is stochastic, and thus, the markup is stochastic, which we assume evolves according to the process:

$$\mathcal{M}_{H,t} = \mathcal{M}_{H,t-1}^{\rho_{\mathcal{M}_H}} \mathcal{M}_H^{1-\rho_{\mathcal{M}_H}} e^{\varepsilon_{\mathcal{M}_H}}$$

where

$$\varepsilon_{\mathcal{M}_H} \sim N(0, \sigma^2_{\mathcal{M}_H}), \quad \rho_{\mathcal{M}_H} \in (0,1)$$

and

$$\mathcal{M}_H = \frac{\epsilon_H}{\epsilon_H - 1} \in \mathcal{R}_+$$

Further, to secure the existence of an optimal price, and rule out unprofitability, we impose that $\epsilon_{H,t} > 1$. Note that as the elasticity of substitution tends to unity from above, the markup tends to infinity.

Near to identically, we get the first order condition with respect to $\mathcal{P}^*_\tau$ as

$$\begin{aligned} 0 = \mathbb{E}_\tau &\left[\sum_{t=\tau}^{\infty} (\theta^*)^{t-\tau} \Delta_{t|\tau,\tau} \xi_t A^*_{H,t|\tau}(i)(1-\epsilon^*_{H,t})\mathcal{P}^*_\tau \left(\frac{P^*_{t-1}}{P^*_{\tau-1}}\right)^{\gamma^*_H}\right] \\ &- \mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} (\theta^*)^{t-\tau} \Delta_{t|\tau,\tau} \xi_t A^*_{H,t|\tau}(i)(1-\epsilon^*_{H,t})\mathcal{M}^*_{H,t} MC^*_{t|\tau}\right] \end{aligned} \tag{34}$$

As we show in Appendix A, section 7.6, every firm faces the same maximisation problem and therefore has the same optimal price. Therefore, we can derive the law of motion for the domestic price index in terms of the current price and the optimal price, and we get

$$P_{H,t} = \left[\theta_H \left(\left(\frac{P_{H,t-1}}{P_{H,t-2}}\right)^{\gamma_H} P_{H,t-1}\right)^{1-\epsilon_{H,t}} + (1-\theta_H)\mathcal{P}_t^{1-\epsilon_{H,t}}\right]^{\frac{1}{1-\epsilon_{H,t}}} \tag{35}$$

[12] Later we will assume that $\Delta_{t,\tau} = \Lambda_{\tau,t}$ but we will use $\Delta_{t,\tau}$ in the derivation for generality and clarity.

[13] See Appendix A, section 7.6 for details.

[14] Defining the markup of a good as the ratio of the price of that good less marginal cost of producing said good to the price yields: $\frac{\mathcal{M}_{H,t}-1}{\mathcal{M}_{H,t}}$.

Similarly, we get the law of motion for the export price index as

$$P_{H,t}^{*}=\left[\theta_{H}^{*}\left(\left(\frac{P_{H,t-1}^{*}}{P_{H,t-2}^{*}}\right)^{\gamma_{H}^{*}}P_{H,t-1}^{*}\right)^{1-\epsilon_{H,t}^{*}}+(1-\theta_{H}^{*})(\mathcal{P}_{t}^{*})^{1-\epsilon_{H,t}^{*}}\right]^{\frac{1}{\epsilon_{H,t}}} \tag{36}$$

We have analogous expressions for goods produced abroad.

2.6.2 Wage setting

We employ the Calvo mechanism to model wage setting, and only concern ourselves with the domestic wage setting. While firms are assumed to set prices, households set wages. We denote the optimal choice of wages at period τ by $\mathcal{W}_\tau$. Households j's wages evolve according to

$$W_{t+1}(i)=\begin{cases}\mathcal{W}_{t+1}(i) & \text{with probability } (1-\theta_W)\\ \left(\frac{P_t}{P_{t-1}}\right)^{\gamma_W}\frac{Z_{t+1}}{Z_t}W_t(i) & \text{with probability } \theta_W\end{cases}$$

The maximisation problem takes the form

$$\max_{\mathcal{W}_\tau}\mathbb{E}_\tau\left[\sum_{t=\tau}^{\infty}\theta_W^{t-\tau}\beta^{t-\tau}\left(\mathcal{U}_t(C_{t|\tau}(j),C_{t-1|\tau},N_{t|\tau}(j)\right)\right] \tag{37}$$

and is presumed to satisfy the relevant constraint. Similarly to the price setting case, the subscript $t \mid \tau$ indicates conditionality with respect to reset wages at time τ. This problem can be shown to have the following first order condition (see Appendix A, section, 7.7 for details)

$$\begin{aligned}0=&\ \mathbb{E}_\tau\left[\sum_{t=0}^{\infty}\theta_W^{t-\tau}\beta^{t-\tau}(\epsilon_{W,t}-1)N_{t|\tau}(j)MU_{C,t|\tau}(j)\frac{\mathcal{W}_\tau}{P_t}\left(\frac{P_{t-1}}{P_{\tau-1}}\right)^{\gamma_H}\frac{Z_t}{Z_\tau}\right]\\ &-\mathbb{E}_\tau\left[\sum_{t=0}^{\infty}\theta_W^{t-\tau}\beta^{t-\tau}(\epsilon_{W,t}-1)N_{t|\tau}(j)MU_{C,t|\tau}(j)\mathcal{M}_{W,t}MRS_{t|\tau}(j)\right]\end{aligned} \tag{38}$$

where $\mathcal{M}_{W,t}=\frac{\epsilon_{W,t}}{\epsilon_{W,t}-1}$ and $MRS_{t|\tau}=-\frac{MU_{N,t|\tau}}{MU_{C,t|\tau}}$ is the marginal rate of substitution at time t, conditional on that wages were last reset at time τ. The process for the wage desired markup is defined analogously to the price markup:

$$\mathcal{M}_{W,t}=\mathcal{M}_{W,t-1}^{\rho_{\mathcal{M}_W}}\mathcal{M}_W^{1-\rho_{\mathcal{M}_W}}e^{\varepsilon_{\mathcal{M}_W}}$$

where

$$\varepsilon_{\mathcal{M}_W}\sim N(0,\sigma_{\mathcal{M}_W}^2),\quad \rho_{\mathcal{M}_W}\in(0,1)$$

and

$$\mathcal{M}_W=\frac{\epsilon_W}{\epsilon_W-1}\in\mathcal{R}_+,\quad \epsilon_W>1$$

The law of motion of the wage index becomes (see Appendix A, section 7.7)

$$W_t = \left[\theta_W \left(\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_W} \frac{Z_t}{Z_{t-1}} W_{t-1}\right) + (1-\theta_W)\mathcal{W}_t\right]^{\frac{1}{1-\epsilon_{W,t}}} \tag{39}$$

2.6.3 Exchange rates and terms of trade

We assume that firms price to market, i.e. that prices are set in the currency of the buyer. Thus the price of a foreign produced good sold domestically evolves with respect to domestic prices and that export prices are rigid with respect to foreign prices. As a consequence, the law of one price does not hold identically, i.e. $P_{H,t} \neq \xi_t P^*_{H,t}$ and $P_{F,t} \neq \xi_t P^*_{F,t}$. This presumption in conjunction with home bias implies that purchasing power parity does not hold in general. We thus define the real exchange rate as the non-trivial quantity

$$S_t = \xi_t \frac{P^*_t}{P_t} \tag{40}$$

where ξ_t is the nominal exchange rate giving the home-currency price of one unit of foreign currency. Lastly, we define the terms of trade as

$$T_t = \frac{P_{F,t}}{\xi_t P^*_{H,t}} \tag{41}$$

2.7 Market equilibrium

Since the final good can be transformed into any variety, i.e. investment, consumption, or public consumption good, one-to-one, it follows that the elasticity of substitution between varieties are the same regardless of the use of the final good. Consequently, we may express demand relations for the aggregate of final goods produced by any firm i with the private consumption demand relations. That is, the demand schedule for consumption holds for public consumption, investment and maintenance of machinery. It follows that the demand schedule relationship holds for the aggregate of final goods.

Aggregate demand for domestically produced good $i \in [0, n]$ is given by

$$Y^d_{H,t}(i) = A_{H,t}(i) + A^*_{H,t}(i) \tag{42}$$

where

$$\begin{aligned} A_{H,t}(i) &= C_{H,t}(i) + I_{H,t}(i) + M_{H,t}(i) + G_{H,t}(i) \\ A^*_{H,t}(i) &= A^*_{E,t}(i) + A^*_{g,t}(i) \\ A^*_{l,t}(i) &= C^*_{l,t}(i) + I^*_{l,t}(i) + M^*_{l,t}(i) + G^*_{l,t}(i) \end{aligned}$$

with $l \in \{E, g\}$. The quantity $M_{H,t}(i)$ represents goods devoted to covering capital utilisation costs, which we can informally think of as maintenance cost. We call $A_{H,t}(i)$ ($A^*_{H,t}(i)$) the domestic (foreign) absorption of good i.[15] It represents the total demand for good i, distinguished by where the demand originates.

[15]We define $A_{F,t}$ and $A^*_{F,t}$ analogously.

Similarly for the foreign economy we have

$$Y^d_{F,t}(i) = A^*_{F,t}(i) + A_{F,t}(i) \tag{43}$$

where

$$\begin{aligned} A^*_{F,t}(i) &= C^*_{F,t}(i) + I^*_{F,t}(i) + M^*_{F,t}(i) + G^*_{F,t}(i) \\ A_{F,t}(i) &= C_{F,t}(i) + I_{F,t}(i) + M_{F,t}(i) + G_{F,t}(i) \end{aligned} \tag{44}$$

2.7.1 Domestically produced goods

Each generic domestic firm produces to satisfy the demand for its good, given its price. Let $Y^h_{g,t}(i)$ represent the production of a given generic domestic firm i for the domestic market and $Y^f_{g,t}(i)$ the production of the same firm for the foreign market. Total production of the firm is thus given by $Y_{g,t}(i) = Y^h_{g,t}(i) + Y^f_{g,t}(i)$. We denote the production of the specialised export firm as $Y_{E,t}(i)$.

We can then define aggregate domestic output per capita as

$$Y_{H,t} = Y^h_{H,t} + EX_t \tag{45}$$

where

$$\begin{aligned} Y^h_{H,t} &\equiv \frac{1}{n}\left(\left(\frac{1}{n}\right)^{\frac{1}{\varepsilon_{H,t}}} \int_0^n Y^h_{g,t}(i)^{\frac{\varepsilon_{H,t}-1}{\varepsilon_{H,t}}} di\right)^{\frac{\varepsilon_{H,t}}{\varepsilon_{H,t}-1}} \\ EX_{g,t} &\equiv \frac{1}{n}\left(\left(\frac{1}{n}\right)^{\frac{1}{\varepsilon^*_{H,t}}} \int_0^n Y^f_{g,t}(i)^{\frac{\varepsilon^*_{H,t}-1}{\varepsilon^*_{H,t}}} di\right)^{\frac{\varepsilon^*_{H,t}}{\varepsilon^*_{H,t}-1}} \\ EX_{E,t} &\equiv \min\left(\frac{1}{n}\int_0^n Y_{E,t}(i)di, \frac{\alpha_E}{1-\alpha_E} EX_{g,t}\right) \end{aligned}$$

Total real export is simply the sum of the two types of exports.

$$EX_t = EX_{g,t} + EX_{E,t}$$

Markets for domestically produced goods are in equilibrium when these 3 markets clear. Using the aforementioned fact on elasticity of substitution of varieties for investment, government consumption and maintenance, and inserting these expressions into the aggregate demand equations we get for domestic demand of domestic goods (see Appendix 7.5 for details)

$$Y^h_{H,t} = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} A_t \tag{46}$$

where

$$A_t = C_t + I_t + M_t + G_t$$

For foreign demand of domestic goods we get

$$EX_{g,t} = \alpha^*(1-\alpha_E)\left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta} \frac{\left(1-\Gamma^*_{H,t} - EX_t \frac{\partial \Gamma^*_{H,t}}{\partial EX_t}\right)^{\eta}}{1-\Gamma^*_{H,t}} A^*_t$$

and

$$EX_{E,t} = \begin{cases} Z_t Z_{E,t} \overline{\mathcal{K}} & \Big| \text{ if } Y_{E,t} < EX_{g,t} \\ \frac{\alpha_E}{1-\alpha_E} EX_{g,t} & \Big| \text{ otherwise} \end{cases}$$

where

$$A^*_t = C^*_t + I^*_t + M^*_t + G^*_t$$

2.7.2 Foreign produced goods

Similarly to the domestic case we insert demand schedules into the aggregate production to arrive at market clearing. Next we let n approach its limit, i.e. $n \to 0$, since the domestic economy is a small open one (see Appendix 7.5 for details). This procedure gives

$$\begin{aligned} Y^d_{F,t}(i) &= A^*_{F,t}(i) + A_{F,t}(i) \\ &= \left(\frac{P^*_{F,t}(i)}{P^*_{F,t}}\right)^{-\varepsilon^*_{F,t}} \left(\frac{P^*_{F,t}}{P^*_t}\right)^{-\eta} A^*_t \end{aligned}$$

where we have used that $1-\bar{\alpha}^* = 1 - n\alpha^*$ and $1-\bar{\alpha} = (1-n)\alpha$. Note that foreign production is independent of domestic demand for foreign goods. Imports per capita are given as

$$IM_t \equiv \frac{1}{n}\left(\left(\frac{1}{1-n}\right)^{\frac{1}{\varepsilon_{F,t}}} \int_n^1 Y^h_{F,t}(i)^{\frac{\varepsilon_{F,t}-1}{\varepsilon_{F,t}}} di\right)^{\frac{\varepsilon_{F,t}}{\varepsilon_{F,t}-1}}$$

In equilibrium we have $Y^h_{F,t}(i) = A_{F,t}(i)$, and thus for imports we get

$$IM_t = \alpha \left(\frac{P_{F,t}}{P_t}\right)^{-\eta} A_t \tag{47}$$

2.8 Net exports and GDP

Net exports per capita in terms of domestically produced goods are defined as

$$NX_t \equiv EX_t - \frac{P_{F,t}}{\mathcal{E}_t P^*_{H,t}} IM_t = EX_t - T_t \cdot IM_t \tag{48}$$

where T_t represents the terms of trade. We define the GDP deflator as

$$P_{Y,t} = P_t P^{-\alpha}_{F,t} (\xi_t P^*_{H,t})^{\alpha} = P_t T_t^{-\alpha}$$

This specification of the GDP deflator aids in identification of the model's parameters since it reduces the likelihood of nominal shocks being interpreted as real shocks. Furthermore, it links the real GDP defined in the model to the expenditure approach definition of GDP in the national accounts.[16] We define nominal gross domestic product per capita as

$$P_{Y,t}GDP_t \equiv P_{H,t}Y_{H,t}^h + \mathcal{E}_t P_{H,t}^* EX_t$$

Real GDP per capita in consumption units is therefore

$$\begin{aligned} Y_t \equiv GDP_t &= \left(\frac{P_{Y,t}}{P_t}\right)^{-1}\left(\frac{P_{H,t}}{P_t}Y_{H,t}^h + \frac{\mathcal{E}_t P_{H,t}^*}{P_t}EX_t\right) && (49)\\ &= \left(\frac{P_{H,t}}{P_{Y,t}}Y_{H,t}^h + \frac{\mathcal{E}_t P_{H,t}^*}{P_{Y,t}}EX_t\right)\\ &= (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{1-\eta} T_t^\alpha A_t + \frac{\mathcal{E}_t P_{H,t}^*}{P_{Y,t}}EX_t && (50)\end{aligned}$$

2.9 Monetary policy

To close the model we define a monetary policy rule dictating interest rates. Monetary authorities are assumed to set rates according to the following augmented Taylor rule:

$$\frac{R_t}{R} = Z_{R,t}\left(\frac{R_{t-1}}{R}\right)^{\xi_R}\left[\left(\frac{\Pi_{P,t}}{\Pi_P}\right)^{\phi_P}\left(\frac{\overline{Y}_t}{Y}\right)^{\phi_Y}\left(\frac{\overline{Y}_t}{\overline{Y}_{t-1}}\right)^{\phi_{\Delta Y}}\right]^{1-\xi_R} \tag{51}$$

where $0 < \xi_R < 1$ governs interest rate inertia and

$$Z_{R,t} = e^{\varepsilon_R}, \quad \varepsilon_R \sim N(0, \sigma_R^2)$$

is a monetary policy shock. By mandate, the Central Bank of Iceland seeks to minimize inflation deviation from inflation target, on average. Thus, in our specification, monetary authorities respond to contemporaneous inflation, output and output growth. The latter two are intended as proxies for future inflation. Our specification of the monetary policy rule deviates from that of earlier versions. DYNIMO I and II followed Adolfson et al. (2007) in including exchange rates and past inflation explicitly in the monetary policy rule. Their specification was influenced by the fact that inflation targeting was legislated in 2001, coming into full effect in 2003, amidst their estimation period of 1991Q1 – 2015Q4.

3 Estimation

We estimate the model using Bayesian techniques, sampling 4 million draws using the Random Walk Metropolis-Hastings algorithm, provided by Dynare (Adjemian et al., 2011).[17] Naturally, the Bayesian approach requires us to decide on prior distributions for the parameters we intend to estimate. We take inspiration from the posterior estimates in Seneca (2010) and Gestsson (2013) when setting priors. As aforementioned, they estimate the model on data series covering the period 1991Q1 – 2015Q4. Our dataset ranges from 2011Q1 to 2019Q4. In doing so we avoid having the collapse of the Icelandic financial sector influence the parameter estimation. This

[16] DYNIMO I and II defined $P_{Y,t} = P_t$.

[17] See Chib and Greenberg (1995) for a detailed explanation of the Metropolis-Hastings algorithm.

was deemed appropriate since the model has no financial sector and has therefore no mechanism to sensibly assign the actualised shocks that hit the economy during that tumultuous period to the model shocks. The estimated model is an augmentation of the model presented in the previous section. We do so to ensure a better fit to the data. In particular, we add a smoothing factor to the UIP, affix a measurement error to the import price observation equation, and append maintenance cost to the model variable I_t when matching it to investment data, i.e. $\mathring{I}_t = I_t + M_t$, where $\mathring{I}_t$ denotes the transformed investment data.

3.1 Data and measurement equations

The base dataset consists of quarterly data from Statistics Iceland. For indices we rely on a database created for CBI's quarterly macroeconomic model, QMM (Daníelsson et al., 2019). For the national account variables we use moving averages to dampen short run shocks, due to e.g. data irregularities, not captured in the model, to curtail the effects such irregularities have on forecasts. When applicable, a time series is seasonally adjusted and irregular components are removed. Time series are listed in table 1.

Table 1: List of data linked model variables.

Description	*Variable*
Gross domestic product	Y_t
Consumption	C_t
Investment	I_t
Government consumption	G_t
Total imports	IM_t
Total exports	EX_t
Generic exports[18]	$EX_{g,t}$
CPI	P_t
Import prices	$P_{F,t}$
GDP deflator	$P_{Y,t}$
Real exchange rate	S_t
CBI's key interest rate	R_t
Trade weighted foreign GDP index	Y_t^*
Trade weighted foreign price level	P_t^*
Trade weighted foreign interest rate	R_t^*

Since the model is specified in deviations from steady state we need to detrend the data. Trends are found in a few different ways. Selected variables are not transformed and we detrend the level. For another group of variables we find the percentage growth of the underlying time series and detrend therefrom. Lastly, a selection of variables is transformed into a ratio of other variables, namely as a ratio of GDP, CPI or total exports. Which method is used depends on what properties of the variable we want to ensure hold when forecasting, what trends are currently in use in QMM, and what information the estimation procedure can extract for the shock path determination. For example, using the growth of GDP to match with the model variable Y_t has direct implications for the labour augmenting technology shock, which in turn affects estimation of other shocks impacting wages and labour. Another illustrative example is that we want to ensure that there isn't a growing wedge between GDP deflator and CPI and we therefore match the GDP deflator with the data as a ratio of CPI. Furthermore, we apply a 3-quarter moving average smoother on the subcomponents of GDP, after seasonal adjustment, but before any other transformation is applied. This is justified on the grounds that Iceland is a small country where frequent idiosyncratic non-modelled shocks find their way into the national accounts. By applying moving average these distortive movements are ironed out. Let X_t^T denote the trend time series for variable X_t. Following is a list of the data transformations employed:

1. $OBS\_X_t = \ln\left(\frac{X_t}{X_{t-1}}\right) - \ln\left(\frac{X_t^T}{X_{t-1}^T}\right)$
2. $OBS\_X_t = \frac{X_t}{X_{0,t}} \frac{X_t^T}{X_{0,t}^T} - 1$
3. $OBS\_X_t = \ln(X_t) - \ln(X_t^T)$
4. $OBS\_X_t = aX_t - bX^T$

We refer to table 10 in Appendix B for details on which method is applied to which variable. We rely on previous estimates for a few of the trends. We follow Pétursson (2018) in choosing the domestic equilibrium inflation rate, Π^{eq}; take note of Daníelsson et al. (2016) when choosing the real natural interest rate, R^{eq}; and refer to the QMM handbook (Daníelsson et al., 2019) for the trade weighted equilibrium foreign inflation rate, $(\Pi^*)^{eq}$ and the equilibrium growth rate of GDP, $ln(\frac{Y_t^T}{Y_{t-1^T}})$.

3.2 Calibration

The purpose of calibration is two-fold: firstly, calibration is necessary to elicit a unique steady-state, and secondly, calibration can mitigate effects of weak identification. We use several source to determine appropriate calibration values. Our biggest inspiration are the two previous versions of DYNIMO by Seneca (2010) and Gestsson (2013), and the CBI's quarterly macroeconomic model QMM (Daníelsson et al., 2019). Additional guidance comes from Riksbank's RAMSES II (Adolfson et al., 2013), Norges bank's NEMO (Kravik and Mimir, 2019), the Federal Reserve of New Zealand's NZSIM (Kamber et al., 2016), and microeconomic results for which there is a reasonable consensus. Lastly, we consider empirical moments over the estimation period. Average GDP component shares and subcomponent ratios for the periods 1991Q1-2005Q4, over which DYNIMO I and DYNIMO II are estimated, and 2011Q1-2019Q4, over which our model is estimated, are shown in table 2. The wedges between the averages in the different periods indicate a structural shift in the Icelandic economy in the wake of the financial crisis. A fact which prompts reconsideration of the calibration values.

Changes in output shares of different industries over time, e.g. tourism and fisheries, are in agreement with the idea that the Icelandic economy is fundamentally different in these periods (see figure 1). This is further supported by the change in the tourism sector's share of GDP, going from roughly 3.5% in 2009 to 8.1% in 2016. The number of yearly visitors per capita grew by 1000% between 1991 and 2018.[20] In addition, the globally persistent low inflation and low interest rate environment has potential implications for the natural interest rate in Iceland. Daníelsson et al.(2016) indeed find that the natural real rate of interest has decreased in recent

[18]Generic export is defined as total export less exports of marine products, aluminium, aeroplanes and ships.
[19]Non-subscripted variables are real variables and variables with the subscript N are nominal.
[20]Source: OECD, Statistics Iceland, Icelandic Tourist Board.

Table 2: Share of GDP components

Ratios[19]	91Q1–05Q4	11Q1–19Q4
$\frac{C}{Y}$	56.0%	50.1%
$\frac{C_N}{Y_N}$	57.1%	52.6%
$\frac{I}{Y}$	21.9%	16.8%
$\frac{I_N}{Y_N}$	23.2%	18.6%
$\frac{G}{Y}$	25.1%	22.1%
$\frac{G_N}{Y_N}$	22.4%	23.8%
$\frac{EX}{Y}$	29.8%	41.9%
$\frac{EX_N}{Y_N}$	33.6%	50.0%
$\frac{IM}{Y}$	33.4%	33.1%
$\frac{IM_N}{Y_N}$	36.4%	45.0%
$\delta = \frac{I}{K}$	1.79%	1.50%
$\alpha_E = \frac{EX_E}{EX}$	56.0%	41.8%

years. Since the parameters of a microfounded DSGE model are considered deep, in the sense that they reflect the structural aspects of the economy, it is necessary as a consequence of the aforementioned reasons to accommodate the perceived structural shift by letting our calibrated values differ from previous versions of DYNIMO.

From steady-state equations (214), (218), and (219), derived in Appendix C, we have

$$\frac{K}{Y} = \frac{(1-\alpha\cdot\alpha_E)\psi_H}{\mathcal{M}_H(\beta^{-1}-1+\delta)} + \alpha\alpha_E \tag{52}$$

$$\frac{I}{Y} = \delta\frac{K}{Y} \tag{53}$$

$$\frac{C}{Y} = 1 - \frac{I}{Y} - \gamma_g \tag{54}$$

where $\frac{G}{Y} = \gamma_g$. The steady-state assumption of vanishing trade balance, $NX = EX - IM = 0$, is not in agreement with the data over the period 2011Q1-2019Q4, where we have seen a persistent trade surplus. Therefore we allow for our calibrated values of the "great ratios"[21] to imply non-conforming steady-state ratios. The ratio of capital to investment, the steady state value of which is δ, has fluctuated in the interval $(0.01, 0.03)$ between 1991Q1-2019Q4 with an average of 0.017. The value decided upon in DYNIMO I and II was $\delta = 0.02$, arguing that the investment share was more plausible with that specification. The argument holds equally well in our case and we follow tradition in setting $\delta = 0.02$. In accordance with QMM (Daníelsson et al., 2019) the capital share is fixed as $\psi_H = 0.4$. For the foreign capital share we fix $\psi_F = 0.33$, as in DYNIMO I and II. From the first order conditions for households, we see that on a balanced growth path, the stochastic discount rate is given by

$$\Lambda_{BGP} = (\Pi_P \Pi_Z)^{-1}\beta$$

Recall that we define the nominal gross interest rate as the inverse of the expected value of the

[21]The "great ratios" term was coined by Klein and Kosobud (1961) and refers to those ratios of macroeconomic variables which are presumed stationary. In our case it refers exclusively to the ratios of consumption, investment, and government consumption to output, as well as the ratio of investment to capital.

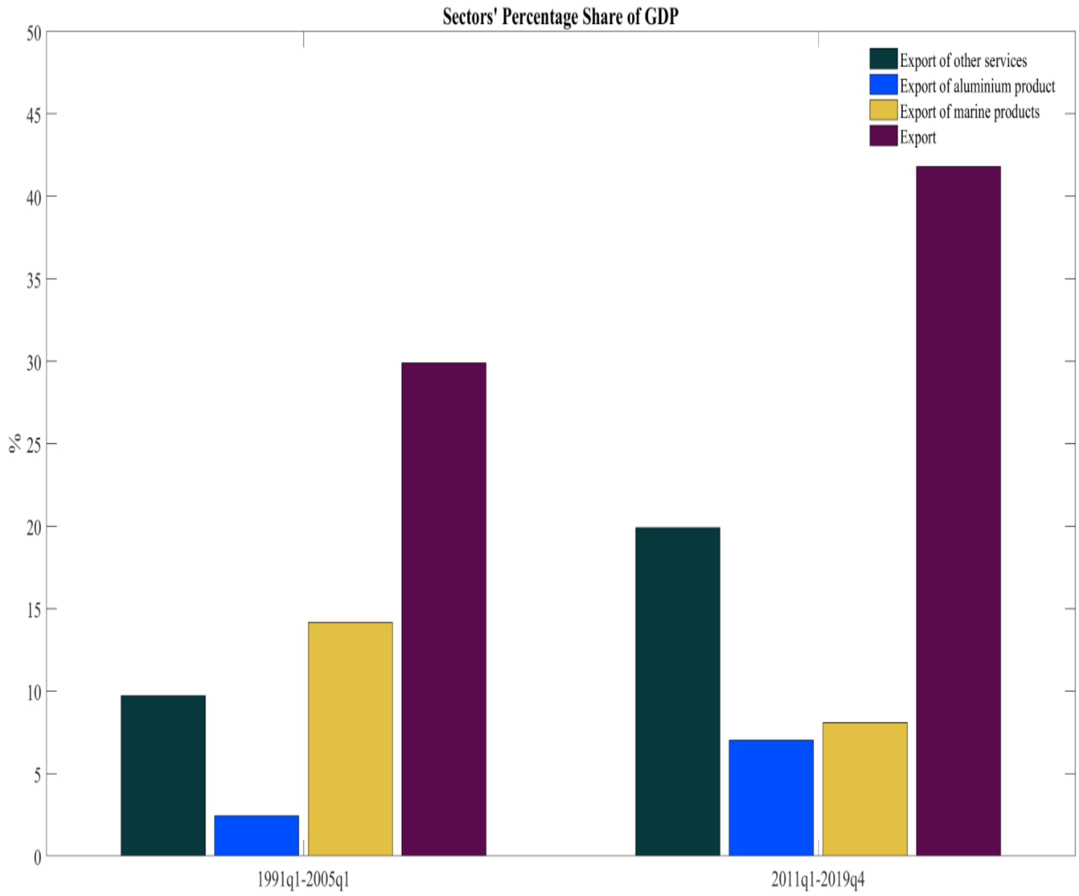

Figure 1: Average percentage share of GDP for different export sectors over the two relevant periods.

stochastic interest rate, and we can thus write the gross natural real rate as

$$\frac{R}{\Pi_P} = \Pi_Z \beta^{-1}$$

Setting $\beta = 0.995$, we arrive at annual steady state interest rates in harmony with Daníelsson et al. (2016). The ratio of government consumption to GDP is calibrated at $\gamma_G = 0.24$. We now get a restriction on the domestic markup from the steady state relations above in conjunction with those given in equations (215) (see Appendix C for details):

$$\mathcal{M}_H = \frac{(1 - \alpha \cdot \alpha_E)\psi_H}{(\beta^{-1} - 1 + \delta)\left(\frac{Y}{K} - \alpha \cdot \alpha_E\right)}$$

Supplementing these restrictions on the markup with moment matching simulation experiments and an appeal to the broad literature we settle on $\epsilon_H = 6$, which implies a markup of 20%. We follow DYNIMO I in regards to equating price elasticity of substitution of domestically produced goods and imported goods and calibrate price elasticity of substitution for imported goods as $\epsilon_F = 6$. Which is in line with RAMSES II and NEMO. For labour, moment matching suggests

ϵ_W on the interval $(3, 5)$. Opting to not deviate too far from previous versions, we set $\epsilon_W = 4.5$. Steady state assumptions and restrictions necessitate

$$\alpha = \frac{EX}{Y} = \frac{IM}{Y}$$

In light of the disparity between these ratios in the data we take the approach of following the export share and calibrate $\alpha = 0.42$.

Following DYNIMO I, where an AR model is estimated to determine the persistence of government consumption and the error variance, we set $\rho_g = 0.8$ $\sigma_g = 0.0051$. We run an additional AR model to determine the persistence in specialised exports, for which we find $\rho_E = 0.93$. In accordance with the argument given in DYNIMO I for calibrating the price and wage indexation parameters in order to reduce the dimensionality of the estimation we set $\gamma_H = 0.75$, $\gamma_F = 0.75$, $\gamma_H^* = 0.75$, $\gamma_W = 0.75$. The values were determined through moment matching and simulations investigating the price response of monetary shocks. Again following DYNIMO I, we calibrate the parameters for international elasticity of substitution, η, the inverse labour supply elasticity, ϕ, and elasticity of capital utilisation, λ_U. Seneca (2010) refers to a discussion in Adolfsson et al. (2007), who argue that international elasticity of substitution is driven to artificially high values by the "estimation procedure's attempt to reconcile the high volatility of imports relative to consumption with the import demand relations," which is a function of total domestic demand and relative prices. Seneca continues: "The nominal rigidities needed to generate plausible responses to monetary shocks, for example, will only allow this relation to add up if η is very high." Values for η vary in the literature. In their report for the European Commission, Imbs and Méjean (2010) find values ranging from 0.5 to 2.7 for 30 different countries. For Sweden, Adolfsson et al. (2013) estimate η as 1.5. Kravik and Mimir (2019) calibrate it as 0.5 for Norway and Kamber et al. (2016) determine it as 0.75 for New Zealand. Previous versions of DYNIMO had $\eta = 4.5$, notably higher than the other references. Taking note of these values and doing simulations, we settled on $\eta = 2$. By similar reasoning, and using the same sources, we set the inverse Frisch elasticity, ϕ, as 3. Complying with Seneca's (2010) argument, we set $\lambda_U = 99999$, which implies a fixed capital utilisation. We also follow DYNIMO I in setting the foreign marginal cost of output factor as $\eta^*_{mc,y} = 2.7$, where Galí et al. (2007) is given as precedent. Following QMM (Daníelsson et al., 2019), we calibrate interest rate smoothing as $\xi_R = 0.6$. The ratio of specialised exports to total export is set as $\alpha_E = 0.36$. Finally, we calibrate export adjustment cost parameter as $\phi^*_M = 5$. Table 3 contains an exhaustive list of calibrated parameters and their values.

Table 3: Calibrated parameters

Parameter	*Value*	*Description*
α	0.42	Openness
β	0.995	Discount factor
δ	0.02	Depreciation rate
ϵ_H	6	ESUB[22]of domestic goods
ϵ_F	6	ESUB of imports
ϵ_W	4.5	ESUB of labour service
η	2	International ESUB
γ_g	0.24	Government spending share
ψ_H	0.4	Capital share of domestic production

(Continued on next page)

Table 3: (continued)

Parameter	*Value*	*Description*
ψ_F	0.33	Capital share of foreign production
$\eta^*_{mc,y}$	2.7	Foreign marginal cost of output factor
γ_H	0.75	Indexing of domestic goods prices
γ_F	0.75	Indexing of imported goods prices
γ^*_H	0.75	Indexing of exported goods prices
γ_W	0.75	Indexing of wages
ρ_G	0.8	Persistence of government spending
ρ_E	0.93	Persistence of exogenous export
ϕ	3	Inverse of labour supply elasticity
λ_U	99999	Elasticity of capital utilisation costs
α_E	0.36	Specialised export share
ξ_R	0.6	Interest rate smoothing
ϕ^*_M	5	Export adjustment cost

3.3 Priors and results

As previously stated we use estimated posteriors from previous versions of DYNIMO for guidance in prior selection. We deviate from them where more agnosticism is viable and choose to use uniform distributions or the β-distribution with shape parameters $(1, 1)$, which has the same probability density function as the uniform distribution on the interval $(0, 1)$. In addition we slightly increase the variance from the reference posteriors in forming our priors to accommodate new information. Furthermore, to facilitate the variance increase we must slightly modify the prior means of those parameters whose posterior means in DYNIMO I and II were estimated close to the edge of their underlying bounded prior distributions. As a rule of thumb the choice of prior distributions is determined by the following criteria: the beta distribution family is chosen when theory predicts that the relevant parameter is on the interval $(0, 1)$, the inverse-gamma distribution is chosen when the parameter are required to be positive, otherwise it is assumed to be normally/uniformly distributed. The decision between using a uniform or normal distribution rests on how informative we want the distribution to be, which in turn is influenced by our opinion on the relevance of the reference posterior of that particular parameter. We make exceptions to the rule of thumb with regards to priors for the standard deviation of shocks which had large posterior means in the reference posteriors. We use a truncated normal distribution for them. Furthermore, we adjust the initial value of the mode estimation for increased efficiency. The prior and posterior distributions of estimated dynamic parameters and shocks' persistence parameters are given in table 4 below. The prior and posterior distributions of the standard deviations of shocks are summarised in table 5. In Appendix D, the priors and posteriors of the parameters from the VAR model of the foreign economy are found. Figures of prior distributions and posterior distributions, as well as of convergence diagnostics, are found in Appendix D.

[22]ESUB stands for elasticity of substitution.

Table 4: Priors and posteriors of dynamic parameters

	Prior			Posterior			
	Dist.	Mean	Stdev.	Mean	Stdev.	HPD inf	HPD sup
h_C	beta	0.700	0.1500	0.941	0.0214	0.9089	0.9744
h_N	beta	0.500	0.1500	0.529	0.1565	0.2754	0.7922
λ_I	invg	0.100	Inf	0.051	0.0198	0.0263	0.0762
ϕ_B	invg	0.010	Inf	0.005	0.0022	0.0024	0.0085
$\phi_{\Delta Y}$	norm	0.150	0.0500	0.134	0.0319	0.0807	0.1856
ϕ_P	norm	1.500	0.2000	1.417	0.1304	1.2059	1.6245
ϕ_Y	norm	0.150	0.0500	0.019	0.0076	0.0068	0.0319
θ_F	beta	0.500	0.1000	0.601	0.0612	0.5009	0.7019
θ_h	beta	0.700	0.1000	0.727	0.1028	0.5751	0.8912
θ_h^*	beta	0.500	0.1000	0.282	0.0726	0.1648	0.3964
θ_w	beta	0.600	0.1000	0.356	0.0759	0.2312	0.4811
ξ_S	beta	0.750	0.1500	0.630	0.0732	0.5120	0.7474
ρ	beta	0.500	0.1500	0.645	0.0665	0.5401	0.7554
ρ_B	beta	0.500	0.1500	0.446	0.1321	0.2253	0.6596
ρ_C	beta	0.500	0.1500	0.820	0.0551	0.7331	0.9086
ρ_D	beta	0.500	0.1500	0.728	0.0688	0.6174	0.8399
ρ_H	beta	0.500	0.1500	0.899	0.0467	0.8290	0.9727
ρ_I	beta	0.500	0.1500	0.707	0.0733	0.5898	0.8289
ρ_{μ_F}	beta	0.500	0.1500	0.639	0.1152	0.4514	0.8271
ρ_{μ_H}	beta	0.500	0.1500	0.599	0.1292	0.3913	0.8176
$\rho_{\mu_H}^*$	beta	0.500	0.1500	0.623	0.1252	0.4218	0.8235
ρ_{μ_W}	beta	0.500	0.1500	0.731	0.0738	0.6119	0.8509

Comparing the results to DYNIMO I and II, we find that the parameters of the UIP are drastically different. Specifically, we estimate the risk premium elasticity as more than 50% smaller than found in DYNIMO II and 75% smaller than found in DYNIMO I. Therefore, the risk premium is not as sensitive to net asset position changes as before. Furthermore, there is significantly less persistence in the risk premium shock in DYNIMO III. Together, this should on average result in higher frequency deviation in the risk premium around a generally smaller value, in absolute terms. This result is unsurprising given the aforementioned metamorphosis of the export sector. In addition, we have added specialised export firms and changed the monetary policy rule, which both affect variables found in the UIP. The former has consequences for net export which influences the exchange rate through the risk premium, while the latter influences the expected interest rate which impacts the real interest rate differential. Disparity can be found in the frequency of wage setting and price setting of export goods. We find that optimal wage contracts last on average 6 months, while in DYNIMO I and II, the average time is found to be close to 12 months. DYNIMO I and II diverge significantly in their estimation in the time between price resets of export goods, with DYNIMO I reporting roughly 4 months between price resets, while DYNIMO II finds it to be closer to 8 months. We find it to be somewhere in the middle, with the average time between price resets being slightly less than 6 months. Another difference between our results and the estimates of previous versions of DYNIMO is that we find shocks to be less persistent. The only exception to this is the persistence of domestic markup, where we find the persistence to be marginally higher.

Table 5: Priors and posteriors of standard deviations of structural shocks.

	Prior			Posterior			
	Dist.	Mean	Stdev.	Mean	Stdev.	HPD inf	HPD sup
σ_G	unif	0.150	0.0866	0.023	0.0030	0.0187	0.0281
σ_Z	unif	0.150	0.0866	0.011	0.0014	0.0091	0.0137
σ_B	unif	0.150	0.0866	0.012	0.0025	0.0080	0.0160
σ_C	norm	0.150	0.0866	0.070	0.0264	0.0319	0.1099
σ_D	unif	0.150	0.0866	0.017	0.0022	0.0137	0.0208
σ_H	unif	0.150	0.0866	0.019	0.0026	0.0148	0.0230
σ_I	norm	0.150	0.0866	0.183	0.0584	0.0866	0.2760
$\sigma_{\mu,F}$	norm	0.150	0.0866	0.077	0.0337	0.0325	0.1241
$\sigma_{\mu,H}$	norm	0.150	0.0866	0.056	0.0430	0.0080	0.1171
$\sigma^*_{\mu,H}$	norm	0.150	0.0866	0.048	0.0170	0.0256	0.0706
$\sigma_{\mu,W}$	norm	0.150	0.0866	0.142	0.0487	0.0660	0.2167
σ_R	unif	0.150	0.0866	0.002	0.0003	0.0014	0.0022
σ_{PF}	unif	0.150	0.0866	0.026	0.0032	0.0203	0.0306
σ_E	unif	0.150	0.0866	0.027	0.0034	0.0217	0.0325
σ^*_π	unif	0.150	0.0866	0.001	0.0001	0.0007	0.0010
σ^*_Y	unif	0.150	0.0866	0.001	0.0002	0.0010	0.0015
σ^*_R	unif	0.150	0.0866	0.000	0.0001	0.0004	0.0006

In general, we find evidence of lower volatility than in DYNIMO I and II. In almost all cases the estimation of standard deviation is lower in our estimate than the previous versions. Specifically, the standard deviations of markup shocks are significantly smaller in DYNIMO III. For example, the standard deviation for domestic markup shock and wage markup shock in DYNIMO II are roughly 4 times larger than we find. A notable exception is the standard deviation for investment technology shock, where we find the standard deviation to be roughly 60% larger than in previous versions. These findings have a significant effect on both forecasting and historical variance decomposition. This follows from the fact that estimation of past states of nature depends upon the standard deviation of shocks and their persistence. Shocks with larger standard deviation are assigned, ceteris paribus, a larger portion of the equilibrium disturbance. Clearly this affects all historical analysis of shock transmission and thus interpretation of causal effects. The results also affect forecasting through historical variance decomposition since a forecast takes the state of nature at the current period as a given and their evolution is then dictated by persistence. This, in turn, continually influences endogenous variables of the model into the future until the shocks die out.

4 Model properties and goodness-of-fit

A simple way to evaluate the goodness-of-fit of a model is to compare moments of model variables to moments of corresponding underlying time series. By construction, the first moments of all variables in the model is 0. Therefore, we turn to the second moments. For a model variable X_t we denote the corresponding observable as $\mathring{X}_t$. Details of transformations are given in Appendix B, section 8.2. First we compare the standard deviations of model variables and the data over the estimation period. Next we consider contemporaneous and first order correlations between model variables and compare them to the correlations found in the data. Following the comparison of second order moments, we calculate impulse response functions and contrast them to that of previous versions of DYNIMO as well as the broad literature. Finally, we calculate the forecast error variance.

4.1 Second order moments

Table 6 compares the standard deviation of model variables to standard deviation of the underlying time series. In the cases of variables denoted $\mathring{X}_t$ we have a direct correspondence between model variables and the input in the estimation. In other cases, we restrict ourselves to the HP filter as a generator of trends, relative to which standard deviations are calculated.

Table 6: Standard deviation of selected variables.

Description	Variables	Model	Data
Output growth	$\mathring{Y}_t$	0.0139	0.0060
Consumption growth	$\mathring{C}_t$	0.0075	0.0063
Investment growth	$\mathring{I}_t$	0.0325	0.0341
Import growth	$\mathring{IM}_t$	0.0214	0.0239
Export share	$\mathring{EX}_t$	0.0691	0.0214
Inflation	$\mathring{\pi}_t$	0.0101	0.0032
Exchange rate growth	$\mathring{S}_t$	0.0369	0.0315
Interest rate	$\mathring{R}_t$	0.0115	0.0022
Hours	$\mathring{N}_t$	0.0441	0.0163
Wage growth	$\mathring{W}_t$	0.0206	0.0102
Output gap	Y_t	0.0827	0.0250
Consumption gap	C_t	0.1289	0.0263
Investment gap	I_t	0.1951	0.0871
Import gap	IM_t	0.1049	0.0917
Export gap	EX_t	0.0894	0.0232

Overall, the model implies more volatility than is found in the data. In most cases, standard deviation of model variables is reasonably close to that of the data. However, in a few cases there is a large disparity. The biggest offenders of the observables are the export share, $\mathring{EX}_t$, the interest rate $\mathring{R}_t$, and inflation $\mathring{\pi}_t$. The source of the volatility is quite possibly the larger standard deviation of the interest rate, which can be explained by our calibrating of the interest rate smoothing parameter in line with QMM (Daníelsson et al., 2019). Comparison between the versions of DYNIMO is made difficult by difference in data handling and the fact that volatility has changed drastically between estimation periods. In general, both the volatility of the model and the data has decreased. There is a shared tendency between all versions of DYNIMO to overestimate volatility. Standard deviation of GDP growth is in all cases roughly two times larger in the model than the data, and standard deviation of inflation is roughly three times larger. For import growth, investment growth and consumption growth, the models and data are relatively harmonious.

4.1.1 Correlations

The problem reported in DYNIMO I that output and imports are countercyclical persists up to first order correlation. However, the second order correlation between import growth and output growth, i.e. $corr(\mathring{IM}_t, \mathring{Y}_{t-2})$, is positive. The reverse correlation becomes positive at order four. With respect to autocorrelation, the model is able to replicate the data well, as seen by inspecting the diagonals in table 8. DYNIMO I and DYNIMO II report contemporary correlation between output growth and other model variables corresponding to observables. In the cases of consumption growth and investment growth our model replicates the data better. The performance is similar for import growth, interest rates and hours. However, the contemporaneous correlation of output growth and inflation matches the underlying data worse than the previous versions of DYNIMO did, and the model is not able to replicate the negative contemporaneous correlation between output growth and inflation. Most plausibly, this is due to the model not considering technology growth to be as dominant a source of output growth as the data suggests and that markup shocks and demand shocks combined are similarly important. Another explanation is that there is a timing discrepancy between model and data, as suggested by the fact that the model produces a negative correlation between output and inflation at orders from one through three. Table 7 depicts the contemporaneous correlation matrix and table 8 shows first order correlation.

4.2 Impulse response functions

Since this version of DYNIMO uses Seneca's (2010) original version as its foundation, propagation and transmission of shocks are largely the same as outlined therein. Therefore, the mechanisms by which shocks propagate through the economy will not be described here in detail. Instead, we will highlight differences and discuss possible sources of incongruence. On the whole, DYNIMO III is less responsive per 1 standard deviation shock. This follows from the fact that we estimate the standard deviations of shocks to be smaller than previous versions. The dissimilitude in monetary policy response rules between versions of DYNIMO is a considerable factor for the differences in impulse responses discussed below. The interest rate smoother parameter is smaller and the relative importance of inflation is larger. This results in nominal variables, e.g. inflation and exchange rate, not being as persistent, which influences the dynamics of international trade. To compensate, the estimation results in more real persistence, and therefore we don't necessarily see consistently faster convergence to equilibrium in DYNIMO III. Rather, it is case specific. Another source of divergence between DYNIMO III and its predecessors is the difference in the UIP. As a result, we find that the dynamics of international trade will relatively often differ between the models. Whether this is a result of a fundamental change in the Icelandic economy due to structural shifts in the export industry as discussed before, a repercussion of the addition of specialised export firms, or a consequence of the difference in the monetary policy rule is

Table 7: Contemporaneous correlation matrix

Model	$\mathring{Y}_t$	$\mathring{C}_t$	$\mathring{I}_t$	$\mathring{IM}_t$	$\mathring{EX}_t$	$\mathring{N}_t$	$\mathring{\pi}_t$	$\mathring{R}_t$
$\mathring{Y}_t$	1							
$\mathring{C}_t$	0.401	1						
$\mathring{I}_t$	0.431	0.227	1					
$\mathring{IM}_t$	-0.254	0.129	0.346	1				
$\mathring{EX}_t$	0.066	0.250	0.251	0.259	1			
$\mathring{N}_t$	-0.012	-0.060	0.043	0.215	0.019	1		
$\mathring{\pi}_t$	0.082	-0.009	0.017	-0.010	0.066	0.293	1	
$\mathring{R}_t$	0.018	-0.047	-0.012	0.023	0.038	0.276	0.890	1
Data								
$\mathring{Y}_t$	1							
$\mathring{C}_t$	0.379	1						
$\mathring{I}_t$	0.329	0.403	1					
$\mathring{IM}_t$	0.395	0.713	0.752	1				
$\mathring{EX}_t$	0.164	0.173	0.518	0.495	1			
$\mathring{N}_t$	0.177	0.573	-0.116	0.224	-0.448	1		
$\mathring{\pi}_t$	-0.333	-0.454	0.141	-0.279	0.538	-0.641	1	
$\mathring{R}_t$	0.130	0.195	-0.348	0.020	-0.610	0.689	-0.701	1

difficult to establish. The impulse responses presented here are broadly similar, qualitatively, to that of previous versions. Thus all comparisons made in Seneca (2010) between DYNIMO I and the relevant literature hold roughly for DYNIMO III, by transitivity. All figures of IRF's are found in Appendix D.

4.2.1 Monetary policy shock

A monetary policy shock is a change in nominal interest rates which can not be explained by the given monetary policy response function. Thus it it can not be anticipated by households via information on inflation, output gap nor the historical path of nominal interest rates. Impulse responses, in percent, to a one standard deviation impulse to monetary policy can be seen in figure 2. The dynamics are broadly similar to the reported IRF's in Seneca (2010). Implying that the transmission mechanism of our model is very much comparable with that of Seneca, and thus the stickiness creating the real interest rate effects, which in turn influence domestic demand is the same. There is, however, a difference in the dynamics of investment and international trade. Investment is substantially slower to bottom out in the new version and net export converges faster. The response of consumption is very dispersed over the horizon, in both DYNIMO I and III, with muted impact and slow convergence. This stems from the fact that in DYNIMO decisions of consumption and investment are based on the expected path of interest rates, as opposed to current interest rates. In addition, habit persistence of consumption is estimated to be high.

Table 8: First order correlations matrix

Model	$\mathring{Y}_{t-1}$	$\mathring{C}_{t-1}$	$\mathring{I}_{t-1}$	$\mathring{IM}_{t-1}$	$\mathring{EX}_{t-1}$	$\mathring{N}_{t-1}$	$\mathring{\pi}_{t-1}$	$\mathring{R}_{t-1}$
$\mathring{Y}_t$	0.541	0.284	0.272	-0.301	0.020	-0.226	-0.017	-0.045
$\mathring{C}_t$	0.368	0.848	0.223	0.114	0.247	-0.079	-0.010	-0.026
$\mathring{I}_t$	0.326	0.186	0.714	0.250	0.276	-0.004	0.019	-0.012
$\mathring{IM}_t$	-0.126	0.153	0.282	0.676	0.205	0.237	0.053	0.053
$\mathring{EX}_t$	0.086	0.239	0.223	0.301	0.973	0.047	0.087	0.056
$\mathring{N}_t$	0.107	-0.039	0.032	0.061	-0.020	0.849	0.199	0.191
$\mathring{\pi}_t$	0.154	0.019	0.021	-0.072	0.037	0.270	0.859	0.718
$\mathring{R}_t$	0.083	-0.038	-0.011	-0.037	0.014	0.299	0.926	0.928
Data								
$\mathring{Y}_t$	0.744	0.305	0.322	0.352	0.074	0.037	-0.357	0.175
$\mathring{C}_t$	0.539	0.857	0.471	0.662	0.160	0.5493	-0.394	0.169
$\mathring{I}_t$	0.260	0.251	0.757	0.444	0.367	-0.077	0.156	-0.245
$\mathring{IM}_t$	0.443	0.657	0.803	0.864	0.401	0.177	-0.232	0.019
$\mathring{EX}_t$	0.322	0.225	0.623	0.542	0.939	-0.487	0.438	-0.637
$\mathring{N}_t$	0.373	0.574	-0.092	0.289	-0.400	0.963	-0.689	0.715
$\mathring{\pi}_t$	-0.251	-0.457	0.013	-0.303	0.539	-0.594	0.800	-0.717
$\mathring{R}_t$	0.114	0.229	-0.366	0.047	-0.482	0.620	-0.603	0.940

4.2.2 Temporary labour augmenting technology shock

In figure 3 the impulse responses to a temporary labour augmenting technology shock is depicted. The dynamics here differ from DYNIMO I, specifically, the dynamics of international trade are different, as well as the response of wages. A labour augmenting technology shock reduces marginal cost, which reduces inflation. The effects of this is twofold. Firstly, it increases domestic consumption and investment, and secondly it increases terms of trade. The second effect, in turn, increasing exports and lessening imports. The latter effect is much more pronounced in the current version than in earlier versions.

4.2.3 Domestic markup shock

A domestic positive markup shock in our model economy can be interpreted as an unanticipated increase in the market power of domestic firms competing in the domestic market or as an increase in the marginal cost of production for domestic firms producing for the domestic market.[23] Impulse responses to a domestic markup shock can be seen in figure 4. A direct consequence of a positive markup shocks is an increase in domestic prices. As a result, consumption is reduced. The central bank reacts to inflation by increasing interest rates, thereby depressing investment. Furthermore, imported goods become relatively more attractive due to the domestic price increase, increasing imports. The combined effect of higher inflation, higher interest rates and increased imports is to push down nominal exchange rates. The nominal exchange rate decrease, in conjunction with a surge in inflation produces real exchange rates depreciation.

[23]The latter interpretation is valid if the change in the marginal cost of production can not be attributed to the total factor productivity shock, the labour augmenting technology shock, or a wage markup shock.

Impulse response dynamics in our model are in line with that of DYNIMO I, although we find less persistence in nominal variables.

4.2.4 Export markup shock

Unlike the case for the domestic markup shock, for the markup shock of exported goods there is a noticeable difference in dynamics of the international market and output between DYNIMO I and III. Although the mechanism is the same, the intensity of the effects are different. Whereas imports fall more than export as a consequence of exchange rate effects of the markup shock in our model, import fall less than export in the previous versions. Therefore, on impact and for 20 quarters, the exported markup shock has the effect of depressing output, despite causing a surge in domestic demand. The impulse response to a export markup shock is shown in figure 5.

4.2.5 Wage markup shock

The dynamics of the response to a wage markup shock are very similar to DYNIMO I. What is interesting to highlight here is the difference between the interest rate response and the persistence of inflationary effects of wage inflation due to a positive wage markup shock. In DYNIMO I, both inflation and interest rates have yet to recover to equilibrium in 20 quarters, while in the current version inflation is close to equilibrium after 7 quarters and interest rates after 9 quarters. Due to the persistent adjustment in the real economy, the path to equilibrium is hump shaped and relatively slow after the initial bounce back.

4.2.6 Risk premium shock

The effects of a more aggressive monetary policy rule is on display in the impulse response to a risk premium shock. Due to lower import prices, inflation deviation from equilibrium is negative on impact. However, after approximately 5 quarters, inflation is nonnegative. In DYNIMO I, inflation deviation is still negative after 10 quarters. Interest rates comparison follows the same script. This in turn affects the real exchange rate through uncovered interest rate parity and consequently impacts net export. The aggressive monetary policy rule discourages investment, which depresses imports. This effect is seen in the original model as well, just with more lag.

4.3 Forecast error variance

We produce forecast error variance decomposition (FEVD) at horizons $h \in \{1, 2, 4, .., 4k\}$ for $k \leq 10$. The FEVD displays the shocks' contribution to the forecasting error variance for a variable of our choice – a conditional historical shock decomposition. Thus, if we desire to evaluate the model, we can see whether the forecasting error is explained by economically plausible shocks. If our aim is to gain insight into the modelled economy, we assume the model correct and the FEVD shows what drives fluctuations in the economy at different horizons. Furthermore, if a shock dominates the FEVD of multiple variables, it can indicate model specification problems or too small a data sample. We will focus on explaining the volatility of interest rates, output, inflation and wages. In addition, we compare our FEVD with the proportion of variance in the historical shock decomposition given in DYNIMO I. We group shocks into foreign, markup, technology, preference, government, and monetary policy shocks.

Monetary policy shock accounts for 30% of the variation of interest rate at 1 quarter in and decreases at longer horizons. Markup shocks dominate early on, with technology shocks becoming similar in importance farther out. A finer categorisation reveals that the markup shock on imported goods, the risk premium shock and the monetary policy shock explain roughly 75% percent of the short to medium term variation.

For output and output growth, technology shocks dominate. Government consumption shocks explains roughly 10% of the variation in output one quarter ahead but decreases to the negligible as the horizon elongates. Markup shocks have the second largest average variation contribution. The permanent technology shock constitutes roughly 60% of the variation in output at a horizon of 20 quarters or longer, while only 25% at a horizon of 1 quarter. Temporary labour augmenting technology shock, asymmetric technology shock and export shocks are other large explanatory factors at short horizons.

Markup shocks explain around 60% of variation in inflation at short horizons, with technology shocks and foreign shocks explaining most of the variation that remains. At longer horizons, technology shocks become more pronounced. The import markup shock has the largest explanatory power of the markup shocks, with domestic markup shock coming in second. With respect to technology shocks, the temporary labour augmenting shock explains 10-15% of the variation over all horizons. Permanent technology shock explains more of the variation at longer horizons, attaining its maximum at 40 quarters with roughly 15%.

Wage inflation variation is dominated at all horizons by the permanent technology shock, comprising roughly 60% of the variation. The second largest explanatory factor being the wage markup shock.

In most cases, the characteristics of the variance contribution matches DYNIMO I. In particular, the two models are concordant with respect to technology explaining by far the largest portion of the variance for output, consumption and investment. In addition, markup is the biggest contributor to fluctuations in interest rates, inflation, and the exchange rate. In contrast, we find that technology contributes to a larger portion of wage inflation as well as for exports and imports , whereas in DYNIMO I markup was the main driver of volatility for these three variables.

An intimately related, but subtly different, method to gauge goodness-of-fit is to calculate the variance contribution of historical shock decomposition. A very similar story is told by that measure. Table 9 shows the largest contributors to volatility for selected variables.

Table 9: Historical shock decomposition contribution in percent.

	Shocks[24]							
	ε_Z	ε_H	ε_D	ε_I	μ_H	μ_W	μ_F	ε_B
$\widehat{Y}_t$	22.8	**29.6**	1.39	13.3	0.74	5.77	0.41	0.44
$\mathring{Y}_t$	**38.5**	23.2	8.21	5.10	3.98	7.73	2.20	2.23
$\widehat{N}_t$	1.42	24.8	9.97	2.53	4.56	**33.2**	1.95	2.62
π_t	3.17	22.5	2.56	0.65	10.6	7.76	**24.3**	14.5
$\pi_{W,t}$	**53.9**	1.65	7.43	2.07	2.72	23.4	1.86	2.53
$\widehat{R}_t$	1.26	16.1	4.13	1.12	4.44	4.34	**27.8**	14.6
$\widehat{S}_t$	3.19	9.63	20.8	3.16	1.65	2.69	**29.0**	16.8

[24]Shock description: ε_Z is a permanent technology shock; ε_H is the temporary labour augmenting technology shock; ε_D is the difference in domestic and foreign technology shock, which can be thought of as a exogenous foreign shock to export demand; ε_I is the investment technology shock; μ_H is the domestic markup shock; μ_W is the wage markup shock; μ_F is the import markup shock; ε_B is the risk premium shock.

5 Percentage deviations from steady-state

All calculation in this subsection rely heavily on methods described in Appendix A, section 7.1.3. The steady state value of a stationary series y_t is given by $\mathbb{E}[y_t] = y$. For a non-stationary series we have an analogous definition for steady state growth. We will sometimes employ the notation $S(y_t) = y$, where S is a projection operator onto the variable's steady state, provided it exists. At times we will abuse notation and use an equality sign instead of the correct approximation sign.

5.1 Steady state assumptions and stationarity conditions

A model seeking to describe the economy at a business cycle frequency must be declarable, directly or indirectly, as a mapping of deviations from steady state.[25] For our definition of deviation to be sensible we require stationarity. Therefore, we must make assumptions on what transformations are necessary to make stationary model variables correspond to their observable counterpart. Furthermore, to solve the model we must presuppose certain relations on our steady states. The nominal variables are assumed non-stationary due to the central bank's non-negative inflation target, which results in a non decreasing evolution of the price level in the long run.[26]

We assume balanced growth path holds for our economy and thus most real variables grow with the permanent total factor productivity shock. Nominal variables naturally grow with the price level. Thus for a domestic real variable X_t, we define $X_t = \overline{X}_t Z_t$. Most domestic nominal variables, X_t, have the representation $X_t = \overline{X}_t P_t$. Analogously, in the foreign case we have $X_t^* = \overline{X}_t^* Z_t^*$ for real variables and $X_t^* = \overline{X}_t^* P_t^*$ for nominal variables. Exceptions to this rule are wages, $W_t = \overline{W}_t P_t Z_t$; capital stock, $K_{S,t} = \overline{K_{S,t}} Z_{t-1}$; marginal utility, $MU_{C,t} = \overline{MU}_{C,t} Z_t^{-1}$; and bonds $B_{H,t} = \overline{B}_{H,t} P_{t-1} Z_{t-1}$, $B_{H,t}^* = \overline{B}_{H,t}^* P_{t-1}^* Z_{t-1}^*$.

Regarding assumptions on the economies' steady state, we suppose that for all exogenous shocks, $Z_{i,t}$, we have $S(Z_{i,t}) = 1$ and assume no adjustment cost in steady state. Further, we assume that $P_H = \Pi_P = \Pi_P^* = \Pi_Z = U = 1$ and $B_{H,t}^* = NX = 0$. Lastly, we impose the condition that

$$\chi = \frac{\overline{W}}{\mathcal{M}_W N^{\phi} \overline{C} (1 - h_N)^{\phi} (1 - h_C)}$$

Steady states of all other variables are derivable from these assumption. Seneca (2010) shows that a unique solution exists to the system of equation that arises. We will thus assume the existence of a unique solution for all steady states and only derive those steady state values that are needed to calculate the percentage deviation from steady state.

[25]This follows from the standard multiplicative decomposition of a time series as

$$X_t = X_t^T S_t C_t I_t$$

where X_t^T is the trend, S_t is the seasonal factor, I_t is the irregular component, and C_t is the cyclical component which we seek to capture. There, of course, exists an additive counterpart to this representation.

[26]This argument holds only if the underlying real variables are sufficiently well behaved. Which here roughly means that they must not be expected to decrease at a rate larger than the price level is expected to increase.

5.2 Production and factor markets

5.2.1 Production

Equation (1) gives the production of a generic domestic firm as

$$Y_{g,t} = K_{g,t}^{\psi_H} (Z_t Z_{H,t} N_{S,t})^{1-\psi_H}$$

where $N_{S,t} = N_t - \hbar N$. Detrended, the equation takes the form

$$\overline{Y}_{g,t} Z_t = (\overline{K}_{g,t} Z_t)^{\psi_H} (Z_t Z_{H,t} N_{S,t})^{1-\psi_H}$$

or equivalently

$$\overline{Y}_{g,t} = \overline{K}_{g,t}^{\psi_H} (Z_{H,t} N_{S,t})^{1-\psi_H}$$

Thus in steady state we have

$$Y_g = K_g^{\psi_H} N_S^{1-\psi_H}$$

and

$$N_S = (1 - \hbar) N$$

where we have used that $Z_H = 1$. To express the detrended equation in percentage deviations from steady state we divide with the steady state and then take the log:

$$\ln\left(\frac{\overline{Y}_{g,t}}{\overline{Y}_g}\right) = \psi_H \ln\left(\frac{\overline{K}_{g,t}}{\overline{K}_g}\right) + (1 - \psi_H)\left(\ln\left(\frac{N_{S,t}}{N_S}\right) + \ln(Z_{H,t})\right)$$

which implies

$$\widehat{Y}_{g,t} = \psi_H \widehat{K}_{g,t} + (1 - \psi_H)(\widehat{N}_{S,t} + \widehat{Z}_{H,t}) \tag{55}$$

In addition we have

$$\widehat{N}_{S,t} = \frac{1}{1-\hbar} \widehat{N}_t \tag{56}$$

Note that if $\hbar = \psi_H$, as in DYNIMO II, we get

$$\widehat{Y}_{g,t} = \psi_H \widehat{K}_{g,t} + \widehat{N}_t + (1 - \psi_H) \widehat{Z}_{H,t}$$

Thus, deviations from steady state in production are a weighted average of the deviation in effective factor inputs, adjusted for overhead labour costs and labour augmenting technology shock.

5.2.2 Marginal cost

The factor input relation from the first order conditions of cost minimisation is given by equation (157):

$$\frac{K_t}{N_{S,t}} = \frac{\psi_H}{1-\psi_H} \frac{W_t}{R_t^K}$$

In a detrended form it becomes

$$\frac{\overline{K}_t}{N_{S,t}} = \frac{\psi_H}{1-\psi_H} \frac{\overline{W}_t}{\overline{R}_t^K}$$

Thus in steady state we have

$$\frac{K}{N_S} = \frac{\psi_H}{1-\psi_H}\frac{W}{R^K}$$

Dividing with the steady state values and taking logs we get

$$\ln\left(\frac{\overline{R}_t^K}{R^K}\right) = \ln\left(\frac{N_{S,t}}{N_S}\right) + \ln\left(\frac{\overline{W}_t}{W}\right) - \ln\left(\frac{\overline{K}_t}{K}\right)$$

or

$$\widehat{R}_t^K = \widehat{N}_{S,t} + \widehat{W}_t - \widehat{K}_t \tag{57}$$

From equation (160) for marginal cost we have

$$MC_t = \frac{1}{1-\psi_H}\left(\frac{\psi_H}{1-\psi_H}\right)^{-\psi_H} \frac{W_t^{1-\psi_H}(R_t^K)^{\psi_H}}{(Z_t Z_{H,t})^{1-\psi_H}}$$

Detrended, the equation takes the form

$$\overline{MC}_t P_t = \frac{1}{1-\psi_H}\left(\frac{\psi_H}{1-\psi_H}\right)^{-\psi_H} \frac{(P_t Z_t \overline{W}_t)^{1-\psi_H}(P_t \overline{R}_t^K)^{\psi_H}}{(Z_t Z_{H,t})^{1-\psi_H}}$$

implying

$$\overline{MC}_t = \frac{1}{1-\psi_H}\left(\frac{\psi_H}{1-\psi_H}\right)^{-\psi_H} \frac{(\overline{W}_t)^{1-\psi_H}(\overline{R}_t^K)^{\psi_H}}{(Z_{H,t})^{1-\psi_H}}$$

Thus in steady state we get

$$MC = \frac{1}{1-\psi_H}\left(\frac{\psi_H}{1-\psi_H}\right)^{-\psi_H} (W)^{1-\psi_H}(R^K)^{\psi_H}$$

To express the detrended equation in percentage deviations from steady state we divide with the steady state values and then apply logarithm on both sides:

$$\ln\left(\frac{\overline{MC}_t}{MC}\right) = (1-\psi_H)\ln\left(\frac{\overline{W}_t}{W}\right) - (1-\psi_H)\ln\left(Z_{H,t}\right) + \psi_H \ln\left(\frac{\overline{R}_t^K}{R^K}\right)$$

which implies

$$\widehat{MC}_t = (1-\psi_H)\widehat{W}_t + \psi_H \widehat{R}_t^K - (1-\psi_H)\widehat{Z}_{H,t} \tag{58}$$

Unsurprisingly, given the determinants of the percentage deviation from steady state for production, the deviation of marginal cost is given as the weighted average of the deviation in effective factor prices.

5.3 Household optimisation

Recall that we define R_t such that

$$1 = R_t \mathbb{E}_t[\Lambda_{t,t+1}]$$

Using equation (142) from Appendix A, we can write the Euler equation as:

$$1 = R_t \mathbb{E}_t[\Lambda_{t,t+1}] = R_t \mathbb{E}_t\left[\beta \frac{Z_{C_{t+1}}(C_{t+1}(j) - h_C C_t)^{-1}}{Z_{C_t}(C_t(j) - h_C C_{t-1})^{-1}}\frac{P_t}{P_{t+1}}\right]$$

and in stationary form

$$1 = R_t \mathbb{E}_t \left[\beta \frac{\Pi_{Z_C,t+1}}{\Pi_{Z,t+1}} \frac{(\overline{C}_{t+1} - h_C \frac{\overline{C}_t}{\Pi_{Z,t+1}})^{-1}}{(\overline{C}_t - h_C \frac{\overline{C}_{t-1}}{\Pi_{Z,t}})^{-1}} \Pi_{P,t+1}^{-1} \right]$$

where $\overline{C}_t(j) = \overline{C}_t$. As Erceg et al. (2000) point out, this follows given the complete markets assumption and, in particular, a complete contingent claims market where all agents share risks. Thus if initial wealth is identical, then so is consumption across households.[27] Using the substitution method we get:

$$\begin{aligned} 0 = \widehat{R}_t + \mathbb{E}_t[\pi_{Z_{C,t+1}}] - \mathbb{E}_t[\pi_{P,t+1}] - \left(\frac{h_C}{1-h_C} + 1 \right) \mathbb{E}_t[\pi_{Z_{t+1}}] \\ - \left(\frac{h_C}{1-h_C} \right) \mathbb{E}_t[\pi_{Z_t}] - \frac{1}{1-h_C} \widehat{C}_{t+1} + \left(\frac{1}{1-h_C} + \frac{h_C}{1-h_C} \right) \widehat{C}_t - \frac{h_C}{1-h_C} \widehat{C}_{t-1} \end{aligned}$$

which we rewrite as

$$\begin{aligned} \widehat{C}_t = \frac{h_C}{1+h_c} \widehat{C}_{t-1} + \frac{1}{1+h_C} \mathbb{E}_t[\widehat{C}_{t+1}] - \frac{1-h_C}{1+h_C} \mathbb{E}_t[\widehat{R}_t - \pi_{P,t+1}] \\ + \frac{1}{1+h_C} \mathbb{E}_t[\pi_{Z,t+1}] - \frac{h_C}{1+h_C} \pi_{Z,t} - \frac{1-h_C}{1+h_C} \mathbb{E}_t[\pi_{Z_C,t+1}] \end{aligned} \tag{59}$$

where

$$\pi_{l,t} = \widehat{\Pi}_{l,t} = \ln(\Pi_{l,t}) - \ln(\Pi_l) = \ln(\Pi_{l,t})$$

Therefore, the deviation in consumption at time t increases with expected deviation in consumption at time $t+1$, as well as past deviations due to habit persistence. Furthermore, real interest rates are the opportunity cost of consumption and thus consumption decreases as a response to an increase in real interest rates. Expected technology growth increases contemporaneous consumption since the households expects to be wealthier next period. Similarly, households adjust consumption downwards if the realisation of a technology shock implies that the household is wealthier than it thought it would be when it decided upon consumption at time $t-1$. Naturally, expectations of consumption preference changes affect consumption.

Uncovered interest rate parity follows from equation (25), which states

$$1 = R_t^*(1-\Gamma_{B,t})\mathbb{E}_t \left[\Lambda_{t,t+1} \frac{\xi_{t+1}}{\xi_t} \right] = R_t^*(1-\Gamma_{B,t})\mathbb{E}_t \left[\Lambda_{t,t+1} \frac{S_{t+1}}{S_t} \frac{\Pi_{P,t+1}}{\Pi^*_{P,t+1}} \right]$$

From the Euler equation, we know that percentage deviation from steady state of the stochastic discount factor equals $-\widehat{R_t}$. Thus, we can write

[27]In their textbook *Foundations for Financial Economics*, Huang and Litzenberger (1988) show that "when the allocation of state contingent claims is efficient and individuals have time-additive state-independent utility functions, prices in the economy are determined as if there were a single individual in the economy endowed with the aggregate endowment."

$$0 = \widehat{R_t^*} - \frac{\Gamma_B}{1-\Gamma_B}\widehat{\Gamma}_{B,t} - \widehat{R}_t + \mathbb{E}_t[\widehat{S}_{t+1}] - \widehat{S}_t + \mathbb{E}_t[\pi_{P,t+1}] - \mathbb{E}_t[\pi^*_{P,t+1}]$$

Let us define $\gamma_{B,t} = \frac{\Gamma_B}{1-\Gamma_B}\widehat{\Gamma}_{B,t}$, and the last equation can be written as

$$\widehat{S}_t = \mathbb{E}_t[\widehat{S}_{t+1}] - (\widehat{R}_t - \mathbb{E}_t[\pi_{P,t+1}]) + (\widehat{R^*}_t - \mathbb{E}_t[\pi^*_{P,t+1}]) - \gamma_{B,t} \tag{60}$$

We assume that $\gamma_{B,t}$ is given by the stochastic process:

$$\gamma_{B,t} = \theta_B b^*_{H,t+1} + z_{B,t} \tag{61}$$

where $b^*_{H,t+1}$ is the home country's real net asset position, and we impose as a proxy

$$b^*_{H,t+1} = \beta^{-1} b^*_{H,t} + \widehat{NX}_t \tag{62}$$

The exchange rate changes as to close risk adjusted arbitrage opportunities. That is, the exchange rate adjusts so that expected risk adjusted profit from strategic purchases of domestic and foreign bonds is zero, the real earning of which are dictated by the corresponding real interest rates.

Finally, we log-linearise the expression related to capital and investment. We begin with the capital utilisation rate, using equation (147) in stationary form:

$$\overline{R}^K = \Gamma'(U_t)$$

Taylor approximation gives

$$\overline{R}^K - R^K = \Gamma''(U)(U_t - U)$$

Using that $U = 1$ and $R^K = \Gamma'(U)$ we get

$$\widehat{R}^K_t = \lambda_U \widehat{U}_t \tag{63}$$

where $\lambda_U = \frac{\Gamma''(1)}{\Gamma'(1)}$. The capital stock and effective capital is related by

$$K_t = U_t K_{S,t}$$

which in stationary form is given by

$$\overline{K}_t = U_t \overline{K}_{S,t} \Pi^{-1}_{Z,t}$$

Elementary operations give the equation in log linear form as

$$\widehat{K}_t = \widehat{U}_t + \widehat{K}_{S,t} - \widehat{\Pi}_{Z,t} \tag{64}$$

The law of motion of capital in stationary form is given by

$$\overline{K}_{S,t+1} = (1-\delta)\overline{K}_{S,t}\Pi^{-1}_{Z,t} + Z_{I,t}\left(1 - \Gamma_I\left(\frac{\overline{I}_t}{\overline{I}_{t-1}}\Pi_{Z,t}\right)\right)\overline{I}_t$$

First we rewrite this equation as

$$1 = (1-\delta)\overline{K}_{S,t}\overline{K}^{-1}_{S,t+1}\Pi^{-1}_{Z,t} + Z_{I,t}\left(1 - \Gamma_I\left(\frac{\overline{I}_t}{\overline{I}_{t-1}}\Pi_{Z,t}\right)\right)\overline{I}_t\overline{K}^{-1}_{S,t+1} \tag{65}$$

Since we assume $\Pi_Z = 1$ and $\Gamma_I(1) = 0$, it is clear that $\delta = \frac{I}{K_S}$. Now using Taylor approximation with respect to $\overline{K}_{S,t+1}, \overline{K}_{S,t}, \overline{I}_t, \overline{I}_{t-1}$, $Z_{I,t}$ and $\Pi_{Z,t}$, we get:

$$\begin{aligned}
0 = (1-\delta)\frac{1}{K_S}(\overline{K}_{S,t} - K_S) - \left((1-\delta)\frac{1}{K_S} + I\frac{1}{K_S^2}\right)(\overline{K}_{S,t+1} - K_S) \\
- (1-\delta)(\Pi_{Z,t} - 1) + \frac{I}{K_S}(Z_{I,t} - 1) + \frac{1}{K_S}(\overline{I}_t - I)
\end{aligned}$$

Using $\delta = \frac{I}{K_S}$, we can rewrite this equation to

$$\widehat{K}_{S,t+1} = (1-\delta)(\widehat{K}_{S,t} - \widehat{\Pi}_{Z,t}) + \delta(\widehat{I}_t + \widehat{Z}_{I,t}) \tag{66}$$

The capital stock increases with past effective investment and decreases in response to past technology shocks. Let us now turn to Tobin's Q. Equation (27) states:

$$Q_t(j) = \mathbb{E}_t\left[\Lambda_{t,t+1}\frac{P_{t+1}}{P_t}\left(\frac{R^K_{t+1}U_{t+1}(j)}{P_{t+1}} - \Gamma_U(U_{t+1}(j)) + (1-\delta)Q_{t+1}(j)\right)\right]$$

Stationary version is given by

$$Q_t(j) = \mathbb{E}_t\left[\Lambda_{t,t+1}\Pi_{P,t+1}\left(\overline{R}^K_{t+1}U_{t+1}(j) - \Gamma_U(U_{t+1}(j)) + (1-\delta)Q_{t+1}(j)\right)\right]$$

and steady state is given by

$$Q = \beta\left(\overline{R}^K + (1-\delta)Q\right)$$

where we use $U = 1$ and by assumption, $\Gamma_U(1) = 0$. Rewriting we get

$$Q = \frac{\beta}{1-(1-\delta)\beta}\overline{R}^K \tag{67}$$

Recalling that $\mathbb{E}_t[\widehat{\Lambda}_{t,t+1}] = -\widehat{R}_t$ and applying Taylor approximation as described in Appendix A, section 7.1.3, gives

$$\widehat{Q}_t(j) = -\mathbb{E}_t\left[\widehat{R}_t + \pi_{t+1}\right] + \frac{\beta R^K}{Q}\mathbb{E}_t\left[\widehat{R}^K_{t+1}\right] + \beta(1-\delta)\mathbb{E}_t\left[\widehat{Q}_{t+1}(j)\right]$$

where we have used that $R = \beta^{-1}$ (see Appendix C for derivation), $\Pi_P = U = 1$ and that $R^K = \Gamma'_U(1)$. Define $\omega_q = \beta(1-\delta)$ and we clearly have $\beta R^K = Q(1-\omega_q)$, and thus we get the desired expression:

$$\widehat{Q}_t(j) = -\mathbb{E}_t\left[\widehat{R}_t + \pi_{t+1}\right] + (1-\omega_q)\mathbb{E}_t\left[\widehat{R}^K_{t+1}\right] + \omega_q\mathbb{E}_t\left[\widehat{Q}_{t+1}(j)\right] \tag{68}$$

Investment decisions of households are dictated by:

$$\begin{aligned}
1 = Z_{I,t}Q_t\left[1 - \Gamma_{I,t} - \Gamma'_{I,t}\cdot\left(\frac{I_t}{I_{t-1}}\right)\right] \\
+ \mathbb{E}_t\left[Z_{I,t+1}Q_{t+1}\Lambda_{t,t+1}\Pi_{P,t+1}\Gamma'_{I,t+1}\cdot\left(\frac{I_{t+1}}{I_t}\right)^2\right]
\end{aligned}$$

which in stationary form is:

$$1 = Z_{I,t}Q_t\left[1-\Gamma_{I,t}\left(\frac{\overline{I}_t}{\overline{I}_{t-1}}\Pi_{Z,t}\right)-\Gamma'_{I,t}\left(\frac{\overline{I}_t}{\overline{I}_{t-1}}\Pi_{Z,t}\right)\cdot\left(\frac{\overline{I}_t}{\overline{I}_{t-1}}\Pi_{Z}\right)\right] \tag{69}$$
$$+\mathbb{E}_t\left[Z_{I,t+1}Q_{t+1}\Lambda_{t,t+1}\Pi_{P,t+1}\Gamma'_{I,t+1}\left(\frac{\overline{I}_{t+1}}{\overline{I}_t}\Pi_{Z,t+1}\right)\cdot\left(\frac{\overline{I}_{t+1}}{\overline{I}_t}\Pi_{Z,t+1}\right)^2\right]$$

For the sake of readability let us, for a moment, disregard the expectation operator. We employ first order Taylor approximation w.r.t. $Z_{I,t}$, $Z_{I,t+1}$, Q_t, Q_{t+1}, I_t, I_{t-1}, I_{t+1}, $\Pi_{Z,t}$, $\Pi_{Z,t+1}$, $\Lambda_{t,t+1}$, $\Pi_{P,t+1}$. Recall that $\Gamma_I(\Pi_Z) = \Gamma'_I(\Pi_Z) = 0$ and $Z_I = \Pi_Z = \Pi_P = 1$. Let f be a function representing the right side of equation (69) and note that

$$\left.\frac{\partial f(\cdot)}{\partial x_t}\right|_{ss} = 0, \quad \text{for } x_t \in \{Z_{I,t+1}, Q_{t+1}, \Pi_{P,t+1}, \Lambda_{t,t+1}\}$$

We thus get

$$\begin{aligned} 0 = {} & Q(Z_{I,t}-1) + (Q_t - Q) + g_1(\cdot)(I_t - I) \\ & + g_2(\cdot)(I_{t-1} - I) + g_3(\cdot)(I_{t+1} - I) \\ & - Q\Gamma''_I(\Pi_{Z,t} - 1) + QR^{-1}\Gamma''_I(\Pi_{Z,t+1} - 1) \end{aligned} \tag{70}$$

where

$$\begin{aligned} g_1(\cdot) &= -Q\Gamma''_I I^{-1} - QR^{-1}\Gamma''_I I^{-1} \\ g_2(\cdot) &= Q\Gamma''_I I^{-1} \\ g_2(\cdot) &= QR^{-1}\Gamma''_I I^{-1} \end{aligned}$$

Recall that $R^{-1} = \beta$ and we rewrite equation (70) as

$$\begin{aligned} 0 = {} & Q\widehat{Z}_{I,t} + Q\widehat{Q}_t - Q\Gamma''_I(1+\beta)\widehat{I}_t \\ & + Q\Gamma''_I\widehat{I}_{t-1} + Q\beta\Gamma''_I\widehat{I}_{t+1} \\ & - Q\Gamma''_I\widehat{\Pi}_{Z,t} + Q\beta\Gamma''_I\widehat{\Pi}_{Z,t+1} \end{aligned}$$

Defining $\lambda_I^{-1} = \Gamma''_I$ and rearranging gives

$$\begin{aligned} (1+\beta)\widehat{I}_t = {} & \lambda_I(\widehat{Z}_{I,t} + \widehat{Q}_t) \\ & + \widehat{I}_{t-1} + \beta\widehat{I}_{t+1} \\ & - \pi_{Z,t} + \beta\widehat{\Pi}_{Z,t+1} \end{aligned}$$

and finally we get

$$\widehat{I}_t = \frac{1}{1+\beta}\left(\beta\mathbb{E}_t[\widehat{I}_{t+1} + \pi_{Z,t+1}] + \widehat{I}_{t-1} - \pi_{Z,t} + \lambda_I(\widehat{Z}_{I,t} + \widehat{Q}_t)\right) \tag{71}$$

Investment increases with past investment and expected investment. It further increases with the expectation of technology growth. By a similar argument as for consumption decisions, investment is adjusted downwards in response to a realisation of larger contemporaneous technology growth shock. Furthermore, investment depends positively on investment augmenting technology shocks. Lastly, by our definition of Tobin's Q as the ratio of the marginal value of capital to the marginal value of consumption, then deviation of investment from its steady state is positively related to the deviation of Tobin's Q.

5.4 Prices, wages and exchange rates

In this section we state the relationships between different price indices and derive the Phillips curves for domestic prices and wages. Export and import prices have analogous representations, which we state, as well as analogous derivations, which we omit. The derivations that follow rely on the general recursive solution stated and proved in Appendix A, section 7.1.5. All Phillips curves have the same form, and a general Phillips curve associated with a good or service x_t is given by

$$\pi_{x,t} = \mathbb{E}_t[\pi_{x,t+1}] + \kappa_x \Theta_{x,t} + \Gamma_{x,t}$$

where $\Theta_{x,t}$ is the cost associated with producing the good or service x_t and $\Gamma_{x,t}$ is the indexing of the price to a related good or service. Thus, the price inflation of x_t increases with expected price inflation of x_{t+1} and with a rise in the cost of production of x_t, assuming that κ_x is positive. This effect is either amplified or dampened by the indexation to the price of a related good, through the term $\Gamma_{x,t}$.

In Appendix A, section 7.4.1, we derive the following representation of the domestic price index

$$P_t = \left[\bar{\alpha} P_{H,t}^{1-\eta} + (1-\bar{\alpha}) P_{F,t}^{1-\eta}\right]^{\frac{1}{1-\eta}}$$

In stationary form this becomes

$$1 = \bar{\alpha} \overline{P}_{H,t}^{1-\eta} + (1-\bar{\alpha}) \overline{P}_{F,t}^{1-\eta}$$

The standard general solution method given in Appendix A, section 7.1.3, yields

$$0 = (1-\alpha)\widehat{P}_{H,t} + \alpha \widehat{P}_{F,t} \tag{72}$$

where we have used that

$$\lim_{n\to 0} 1 - \overline{\alpha} = \lim_{n\to 0} (1-n)\alpha = \alpha$$

For an arbitrary price index X_t,[28] we defined $\Pi_{X,t} = \frac{X_t}{X_{t-1}}$. By assumption we have $\overline{X}_t = \frac{X_t}{P_t}$. Easy algebra gives

$$\ln(\overline{X}_t) - \ln(\overline{X}_{t-1}) = \pi_{X,t} - \pi_{P,t}$$

where $\pi_{X,t} = \ln(\Pi_{X,t})$ and $\pi_{P,t} = \ln(\Pi_{P,t})$. As shown in Appendix C, X is finite, in fact $\ln(X) = 0$, and thus we get

$$\pi_{X,t} = \widehat{X}_t - \widehat{X}_{t-1} + \pi_{P,t} \tag{73}$$

For wages a similar argument shows that we have

$$\pi_{W,t} = \widehat{W}_t - \widehat{W}_{t-1} + \pi_{P,t} + \pi_{Z,t} \tag{74}$$

[28] Be it $P_{H,t}$, $P_{F,t}$ or $P_{H,t}^*$.

5.4.1 Domestic prices

The first order condition for optimal price choice at time τ, $\mathcal{P}_\tau$, for firm i is given by equation (33)

$$0 = \mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \theta_H^{t-\tau} \Delta_{t|\tau,\tau} A_{H,t|\tau} (1-\epsilon_{H,t}) \left(\mathcal{P}_\tau \left(\frac{P_{t-1}}{P_{\tau-1}} \right)^{\gamma_h} - \mathcal{M}_{H,t} MC_{t|\tau} \right) \right]$$

We have that $A_{H,t|\tau} = \overline{A}_{H,t|\tau} Z_t$; $\Pi^{\gamma_h}_{H,(\tau-1,t-1)} = \overline{\Pi}^{\gamma_h}_{H,(\tau-1,t-1)}$; $MC_{H,t|\tau} = \overline{MC}_{H,t|\tau} P_t$; $\mathcal{P}_\tau = \overline{\mathcal{P}}_\tau P_\tau$. Further, we define $\nu_{\tau,t|\tau} \cdot \beta^{t-\tau} = \Lambda_{\tau,t|\tau}$, and get: $\nu_{\tau,t|\tau} = \overline{\nu}_{\tau,t|\tau} \cdot \left(\frac{Z_\tau P_\tau}{Z_t P_t} \right)$. In what follows we assume that $\Delta_{t|\tau,t} = \Lambda_{\tau,t|\tau}$. We can thus write

$$\begin{aligned}
&\mathbb{E}_\tau \left[\sum (\theta_H \beta)^{t-\tau} \overline{\nu}_{\tau,t|\tau} \cdot \left(\frac{Z_\tau P_\tau}{Z_t P_t} \right) \overline{A}_{H,t|\tau} Z_t (1-\epsilon_{H,t}) \overline{\mathcal{P}}_\tau P_\tau \Pi^{\gamma_h}_{H,(\tau-1,t-1)} \right] \\
=&\mathbb{E}_\tau \left[\sum (\theta_H \beta)^{t-\tau} \overline{\nu}_{\tau,t|\tau} \cdot \left(\frac{Z_\tau P_\tau}{Z_t P_t} \right) \overline{A}_{H,t|\tau} Z_t (1-\epsilon_{H,t}) \mathcal{M}_{H,t} \overline{MC}_{H,t|\tau} P_t \right]
\end{aligned}$$

Divide both sides with $Z_\tau P_\tau$, and we get

$$\begin{aligned}
&\mathbb{E}_\tau \left[\sum (\theta_H \beta)^{t-\tau} \overline{\nu}_{\tau,t|\tau} \Pi_{P,(\tau,t)} \overline{A}_{H,t|\tau} (1-\epsilon_{H,t}) \overline{\mathcal{P}}_\tau \Pi^{\gamma_h}_{H,(\tau-1,t-1)} \right] \\
=&\mathbb{E}_\tau \left[\sum (\theta_H \beta)^{t-\tau} \overline{\nu}_{\tau,t|\tau} \Pi_{P,(\tau,t)} \overline{A}_{H,t|\tau} (1-\epsilon_{H,t}) \mathcal{M}_{H,t} \overline{MC}_{H,t|\tau} \Pi_{P,(\tau,t)} \right]
\end{aligned}$$

Now we use the identity $x_t = x(1+\widehat{x}_t)$ and the fact that, approximately, $\widehat{x}_t \widehat{y}_t \approx 0$. Finally, we approximate around a zero inflation steady state. It is readily verified that

$$S\left(\nu_{\tau,t|\tau}\right) = S\left(\Pi_{H,t}\right) = S\left(\Pi_{P,t}\right) = 1$$

and we write

$$\begin{aligned}
&\mathbb{E}_\tau \left[\sum (\theta_H \beta)^{t-\tau} A_H (1-\epsilon_H) \mathcal{P} \cdot Y_{1,t}(\cdot) \right] \\
\approx&\mathbb{E}_\tau \left[\sum (\theta_H \beta)^{t-\tau} A_H (1-\epsilon_H) \mathcal{M}_{\mathcal{H}} MC_H \cdot Y_{2,t}(\cdot) \right]
\end{aligned}$$

where

$$Y_{1,t}(\cdot) = (1 + \widehat{\nu_{\tau,t|\tau}} + \pi_{P,(\tau,t)} + \widehat{A_{H,t|\tau}} + (\widehat{1-\epsilon_{H,t}}) + \widehat{\mathcal{P}_\tau} + \gamma_H \pi_{H,(\tau-1,t-1)})$$

and

$$Y_{2,t}(\cdot) = \left(1 + \widehat{\nu_{\tau,t|\tau}} + \pi_{P,(\tau,t)} + \widehat{A_{H,t|\tau}} + (\widehat{1-\epsilon_{H,t}}) + \mu_{H,t} + \widehat{MC_{H,t|\tau}} + \pi_{P,(\tau,t)} \right)$$

where we have written $\mu_{H,t} = \widehat{\mathcal{M}}_{H,t}$. In steady state $S(Y_{1,t}) = S(Y_{2,t}) = 1$ and we clearly have that in equilibrium

$$\mathcal{P} = \mathcal{M}_H MC_H \tag{75}$$

Cancelling terms gives:

$$\begin{aligned}
&\mathbb{E}_\tau\left[\sum_{t=\tau}^{+\infty}(\theta_H\beta)^{t-\tau}\left(\widehat{\mathcal{P}}_\tau+\gamma_H\pi_{H,(\tau-1,t-1)}\right)\right]\\
\approx&\mathbb{E}_\tau\left[\sum_{t=\tau}^{+\infty}(\theta_H\beta)^{t-\tau}\left(\mu_{H,t}+\widehat{MC_{H,t|\tau}}+\pi_{P,(\tau,t)}\right)\right]
\end{aligned}$$

Since $\widehat{\mathcal{P}}_\tau$ is independent of t we can write

$$\frac{\widehat{\mathcal{P}}_\tau}{(1-\theta_H\beta)}\approx\mathbb{E}_\tau\left[\sum_{t=\tau}^{+\infty}(\theta_H\beta)^{t-\tau}\left(\mu_{H,t}+\widehat{MC_{H,t|\tau}}+\pi_{P,(\tau,t)}-\gamma_H\pi_{H,(\tau-1,t-1)}\right)\right]$$

We can apply the general recursive solution derived in Appendix A, section 7.1.5, given by

$$y_\tau=x_\tau+\omega\mathbb{E}_\tau\left(y_{\tau+1}\right)+\frac{\omega}{1-\omega}\mathbb{E}_\tau z_{\tau,\tau+1}$$

with

$$\begin{aligned}
y_\tau&=\frac{\widehat{\mathcal{P}}_\tau}{(1-\theta\beta)}\\
x_t&=\mu_{H,t}+\widehat{MC_{H,t|\tau}}\\
z_{(\tau,t)}&=\pi_{P,(\tau,t)}-\gamma_H\pi_{H,(\tau-1,t-1)}\\
\omega&=\theta_H\beta
\end{aligned}$$

and we get

$$\begin{aligned}
\frac{\widehat{\mathcal{P}}_\tau}{(1-\theta_H\beta)}=&\left(\mu_{H,\tau}+\widehat{MC_{H,\tau|\tau}}\right)+\frac{\theta_H\beta}{(1-\theta\beta)}\mathbb{E}_\tau\left[\widehat{\mathcal{P}}_{\tau+1}\right]\\
&+\frac{\theta_H\beta}{1-\theta\beta}\mathbb{E}_\tau\left[\pi_{P,(\tau,\tau+1)}-\gamma_H\pi_{H,(\tau-1,\tau)}\right]
\end{aligned}\tag{76}$$

or equivalently

$$\begin{aligned}
\widehat{\mathcal{P}}_\tau=&(1-\theta_H\beta)\left(\mu_{H,\tau}+\widehat{MC_{H,\tau}}\right)+\theta_H\beta\widehat{\mathcal{P}}_{\tau+1}\\
&+\theta_H\beta(\mathbb{E}_\tau[\pi_{P,(\tau,\tau+1)}]-\gamma_H\pi_{H,\tau})
\end{aligned}\tag{77}$$

Now turning to the law of motion for domestic prices, we have from equation (35) that

$$P_{H,t}=\left[\theta_H\left(\left(\frac{P_{H,t-1}}{P_{H,t-2}}\right)^{\gamma_H}P_{H,t-1}\right)^{1-\epsilon_{H,t}}+(1-\theta_H)\mathcal{P}_{H,t}^{1-\epsilon_{H,t}}\right]^{\frac{1}{1-\epsilon_{H,t}}}$$

Detrending we have

$$\overline{P}_{H,t}^{1-\epsilon_{H,t}}=\theta_H\left(\left(\frac{\overline{P}_{H,t-1}}{\overline{P}_{H,t-2}}\right)^{\gamma_H}\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_H}\left(\frac{P_{t-1}}{P_t}\right)\overline{P}_{H,t-1}\right)^{1-\epsilon_{H,t}}+(1-\theta_H)\overline{\mathcal{P}}_{H,t}^{1-\epsilon_{H,t}}$$

Using Taylor approximation w.r.t. $P_{H,t-1}$,$P_{H,t-2}$, $\Pi_{P,t-1}$, $\Pi_{P,t-2}$ and $\epsilon_{H,t}$, we get

$$\widehat{P}_{H,t} = \theta_H(\gamma_H \pi_{H,t-1} + \widehat{P}_{H,t-1} - \pi_{P,t}) + (1-\theta_H)\widehat{\mathcal{P}}_{H,t} \tag{78}$$

which we rewrite as

$$\widehat{\mathcal{P}}_t = \frac{1}{1-\theta_H}\left(\widehat{P}_{H,t} - \theta_H(\widehat{P}_{H,t-1} - \pi_{P,t} + \gamma_H\pi_{H,t-1})\right) \tag{79}$$

For the sake of clarity we suppress the expectation operator in what follows, change the notation to $p_{H,t} = \widehat{P}_{H,t}$, $mc_{H,t} = \widehat{MC}_{H,t}$, and write $\pi_{P,t} = \pi_{P,(t-1,t)}$, $\pi_{H,t} = \pi_{H,(t-1,t)}$. Furthermore, we allow a slight abuse of notation and solve in terms of t rather than τ. Inserting equation (79) into equation (77), we get

$$\begin{aligned}
p_{H,t} - \theta_H(p_{H,t-1} - \pi_{P,t} + \gamma_H\pi_{P,t-1}) = {} & (1-\theta_H)(1-\theta_H\beta)(\mu_t + mc_{H,t}) \\
& + \theta_H\beta p_{H,t+1} \\
& - \theta_H^2\beta\left(p_{H,t} - \pi_{P,t+1} + \gamma_H\pi_{P,t}\right) \\
& + (1-\theta_H)\theta_H\beta(\pi_{P,t+1} - \gamma_H\pi_{P,t})
\end{aligned}$$

Add $\theta_H p_{H,t}$ to both sides and isolate $p_{H,t} - p_{H,t-1} + \pi_{P,t} = \pi_{H,t}$ on the left side to get

$$\begin{aligned}
\pi_{H,t} = {} & \frac{\theta_H - 1}{\theta_H}p_{H,t} + \gamma_H\pi_{P,t-1} \\
& + \kappa(\mu_t + mc_{H,t}) + \beta p_{H,t+1} \\
& - \beta\theta_H p_{H,t} + \beta\theta_H\pi_{P,t+1} - \beta\theta_H\gamma_H\pi_{P,t} \\
& + (1-\theta_H)\beta(\pi_{P,t+1} - \gamma_H\pi_{P,t})
\end{aligned}$$

where $\kappa = \frac{(1-\theta_H)(1-\theta_H\beta)}{\theta_H}$. Now add $\beta p_{H,t} - \beta p_{H,t} + \beta\pi_{P,t+1} - \beta\pi_{P,t+1} = 0$ on the right side to get

$$\begin{aligned}
\pi_{H,t} = {} & \frac{\theta_H - 1}{\theta_H}p_{H,t} + \gamma_H\pi_{P,t-1} \\
& + \kappa(\mu_t + mc_{H,t}) + \beta\pi_{H,t+1} + \beta p_{H,t} - \beta\pi_{P,t+1} \\
& - \beta\theta_H p_{H,t} + \beta\theta_H\pi_{P,t+1} - \beta\theta_H\gamma_H\pi_{P,t} \\
& + (1-\theta_H)\beta(\pi_{P,t+1} - \gamma_H\pi_{P,t})
\end{aligned}$$

Add and subtract $\kappa p_{H,t}$ from the right side and we now have

$$\begin{aligned}
\pi_{H,t} = {} & \frac{\theta_H - 1}{\theta_H}p_{H,t} + \gamma_H\pi_{P,t-1} \\
& + \kappa(\mu_t + mc_{H,t} - p_{H,t}) + \beta\pi_{H,t+1} + \beta p_{H,t} - \beta\pi_{P,t+1} \\
& - \beta\theta_H p_{H,t} + \beta\theta_H\pi_{P,t+1} - \beta\theta_H\gamma_H\pi_{P,t} \\
& + (1-\theta_H)\beta(\pi_{P,t+1} - \gamma_H\pi_{P,t}) + \kappa p_{H,t}
\end{aligned}$$

Define

$$\begin{aligned}
y = {} & \frac{\theta_H - 1}{\theta_H}p_{H,t} + \beta p_{H,t} - \beta\pi_{P,t+1} - \beta\theta_H p_{H,t} + \beta\theta_H\pi_{P,t+1} - \beta\theta_H\gamma_H\pi_{P,t} \\
& + (1-\theta_H)\beta\pi_{P,t+1} - (1-\theta_H)\beta\gamma_H\pi_{P,t} + \kappa p_{H,t} + \beta\gamma_H\pi_{P,t}
\end{aligned}$$

and we can write

$$\pi_{H,t} = \kappa(\mu_t + mc_{H,t} - p_{H,t}) + \beta\pi_{H,t+1} + \gamma_H(\pi_{P,t-1} - \beta\pi_{P,t}) + y \tag{80}$$

Collecting terms in the expression for y:

$$\begin{aligned} y =& p_{H,t}\left(\frac{\theta_H - 1}{\theta_H} + \beta - \beta\theta_H + \kappa\right) + \pi_{P,t}\left(\beta\gamma_h - \beta\theta_H\gamma_H - (1-\theta_H)\beta\gamma_H\right) \\ &+ \pi_{P,t+1}\left(\beta\theta_H - \beta + (1-\theta_H)\beta\right) \end{aligned}$$

Clearly $\beta\theta_H - \beta + (1-\theta_H)\beta = 0$ and $\beta\gamma_h - \beta\theta_H\gamma_H - (1-\theta_H)\beta\gamma_H = 0$ and thus

$$y = p_{H,t}\left(\frac{\theta_H - 1}{\theta_H} + \beta - \beta\theta_H + \kappa\right)$$

We have

$$\begin{aligned} \frac{\theta_H - 1}{\theta_H} + \beta - \beta\theta_H + \kappa &= \frac{\theta_H - 1}{\theta_H} + \beta - \beta\theta_H + \frac{(1-\theta_H)(1-\theta_H\beta)}{\theta_H} \\ &= \frac{\theta_H - 1}{\theta_H}\left(-1 + 1 - \theta_H\beta)\right) + \beta - \beta\theta_H \\ &= (\theta_H - 1)\beta + \beta - \beta\theta_H \\ &= (\theta_H - 1)\beta - (\theta_H - 1)\beta \\ &= 0 \end{aligned}$$

We conclude $y = 0$ and we can write

$$\pi_{H,t} = \kappa(\mu_t + mc_{H,t} - p_{H,t}) + \beta\pi_{H,t+1} + \gamma_H(\pi_{P,t-1} - \beta\pi_{P,t})$$

Bringing back our standard notation and the expectations operator, this implies

$$\pi_{H,t} = \kappa(\mu_t + \widehat{MC}_{H,t} - \widehat{P}_{H,t}) + \beta\mathbb{E}_t[\pi_{H,t+1}] + \gamma_H(\pi_{P,t-1} - \beta\pi_{P,t}) \tag{81}$$

which is the New-Keynesian Phillips curve for domestic prices.

5.4.2 Wages

Let the subscript $t \mid \tau$ for an arbitrary variable denote its value at time t conditional on that the last wage decision was at time τ. Then the first order condition for wage decision $\mathcal{W}$ at time τ of household j is given by equation (38):

$$\begin{aligned} 0 =& \mathbb{E}_\tau\left[\sum_{t=\tau}^{\infty}\theta_W^{t-\tau}\beta^{t-\tau}(\epsilon_{W,t} - 1)N_{t|\tau}(j)MU_{C,t|\tau}(j)\frac{\mathcal{W}_\tau(j)}{P_t}\left(\frac{P_{t-1}}{P_{\tau-1}}\right)^{\gamma_W}\frac{Z_t}{Z_\tau}\right] \\ &- \mathbb{E}_\tau\left[\sum_{t=\tau}^{\infty}\theta_W^{t-\tau}\beta^{t-\tau}(\epsilon_{W,t} - 1)N_{t|\tau}(j)MU_{C,t|\tau}(j)\mathcal{M}_{W,t}MRS_{t|\tau}(j)\right] \end{aligned} \tag{82}$$

where $\mathcal{M}_{W,t} = \frac{\epsilon_{W,t}}{\epsilon_{W,t} - 1}$. We have $\overline{\mathcal{W}}_\tau(j)P_\tau Z_\tau = \mathcal{W}_\tau(j)$ and $MU_{C,t}(j)Z_t = \overline{MU}_{C,t}(j)$, and thus expressing all variables in stationary form gives

$$\begin{aligned} &\mathbb{E}_\tau\left[\sum_{t=\tau}^{\infty}(\theta_W\beta)^{t-\tau}(1-\epsilon_{W,t})N_{t|\tau}(j)\overline{MU}_{C_{t|\tau}}(j)\left(\overline{\mathcal{W}}_\tau(j)\Pi^{-1}_{P,(\tau,t)}\Pi^{\gamma_W}_{P,(\tau-1,t-1)}\right)\right] \\ =&\mathbb{E}_\tau\left[\sum_{t=\tau}^{\infty}(\theta_W\beta)^{t-\tau}(1-\epsilon_{W,t})N_{t|\tau}(j)\overline{MU}_{C_{t|\tau}}(j)\left(\mathcal{M}_{W,t}\overline{MRS}_{t|\tau}(j)\right)\right] \end{aligned}$$

Recall that we defined $\overline{MRS}_{t|\tau}(j) = -\frac{MU_{N,t|\tau}(j)}{\overline{MU}_{C,t|\tau}(j)}$. In the case of flexible wages, i.e. $\theta_W = 0$, it is clear that

$$\mathcal{W}_\tau(j) = \overline{\mathcal{M}}_{W,\tau}\overline{MRS}_\tau(j)$$

That is, households revise real wages at every period as a markup over the rate of marginal substitution between labour and consumption, as we would expect. Assuming $\Pi_P = 1$, the exact same logic and methods painstakingly employed for domestic price decisions, give

$$\frac{\widehat{\mathcal{W}}_\tau(j)}{1-\theta_W\beta} \approx \mathbb{E}_\tau \sum_{t=\tau}^{\infty} (\theta_W\beta)^{t-\tau}(\widehat{\mathcal{M}}_{W,t} + \widehat{MRS}_{t|\tau}(j) + \pi_{P,(\tau,t)} - \gamma_W \pi_{P,(\tau-1,t-1)}) \tag{83}$$

We seek an expression for the optimal wage decision that doesn't depend on the timing of the last wage decision explicitly. Furthermore, we want to express the marginal rate of substitution of household j in terms of the average marginal rate of substitution of the economy. Appealing to our assumption of complete markets and the implied perfect risk sharing,[29] we do not trouble ourselves with effects on consumption. We can write

$$C_{t|\tau}(j) = C_t(j) = C_t$$

The general detrended marginal rate of substitution for household j at time t is given by

$$\overline{MRS}_t(j) = -\frac{MU_{N,t}(j)}{\overline{MU}_{C,t}(j)}$$

where

$$\overline{MU}_{C,t}(j) = Z_{C,t}\left(\overline{C}_t(j) - h_C\frac{\overline{C}_{t-1}}{\Pi_{Z,t}}\right)^{-1} \tag{84}$$

$$MU_{N,t} = -\chi\left(N_t(j) - h_N N_{t-1}\right)^{\varphi} \tag{85}$$

which implies

$$\begin{aligned}\widehat{MRS}_t(j) = {} & \frac{\varphi}{1-h_N}(\widehat{N}_t(j) - h_N\widehat{N}_{t-1}) - \widehat{Z}_{C,t} \\ & + \frac{1}{1-h_C}(\widehat{C}_t(j) - h_C\widehat{C}_{t-1}) + \frac{h_C}{1-h_C}\pi_{Z,t}\end{aligned} \tag{86}$$

For an agent j at time t, conditional on wage reset at time τ, we have a similar expression under our aforementioned assumption of complete markets and the resulting risk sharing:

$$\begin{aligned}\widehat{MRS}_{t|\tau}(j) = {} & \frac{\varphi}{1-h_N}(\widehat{N}_{t|\tau}(j) - h_N\widehat{N}_{t-1|\tau}) - \widehat{Z}_{C,t} \\ & + \frac{1}{1-h_C}(\widehat{C}_t - h_C\widehat{C}_{t-1}) + \frac{h_C}{1-h_C}\pi_{Z,t}\end{aligned} \tag{87}$$

Furthermore, from the expression for conditional labour demand stated in equation (205), we get

$$\widehat{N}_{t|\tau}(j) = -\epsilon_W\left(\widehat{W}_{t|\tau}(j) - \widehat{W}_{t|\tau}\right) + N_{t|\tau} \tag{88}$$

[29]See discussion in section 5.3

and we have

$$\widehat{W}_{t|\tau}(j) = \widehat{\mathcal{W}}_\tau - \pi_{P,(\tau,t)} + \gamma_W \pi_{P,(\tau-1,t-1)} \tag{89}$$

Therefore we can write

$$\widehat{N}_{t|\tau}(j) = -\epsilon_W \left(\widehat{\mathcal{W}}_\tau - \pi_{P,(\tau,t)} + \gamma_W \pi_{P,(\tau-1,t-1)} - \widehat{W}_{t|\tau} \right) + N_{t|\tau} \tag{90}$$

The law of motion of wages takes into account the proportion of the population which can decide upon their wages and is independent of which households in particular reset their wages. Aggregate labour demand depends on aggregate wages. Together this implies for all $t > \tau$

$$W_{t|\tau} = W_t \quad \text{and} \quad N_{t|\tau} = N_t$$

We define the average marginal rate of substitution between labour and consumption as

$$\begin{aligned} \widehat{MRS}_t = & \frac{\varphi}{1-h_N}(\widehat{N}_t - h_N \widehat{N}_{t-1}) - \widehat{Z}_{C,t} \\ & + \frac{1}{1-h_C}(\widehat{C}_t - h_C \widehat{C}_{t-1}) + \frac{h_C}{1-h_C}\pi_{Z,t} \end{aligned} \tag{91}$$

We end up with the following expression

$$\widehat{MRS}_{t|\tau}(j) - \widehat{MRS}_t = -\frac{\epsilon_W \varphi}{1-h_N}\left[\widehat{\mathcal{W}}_\tau - \pi_{P,(\tau,t)} + \gamma_W \pi_{P,(\tau-1,t-1)} - \widehat{W}_t \right] \tag{92}$$

Equation (83) thus becomes

$$\begin{aligned} \nu_W \widehat{\mathcal{W}}_\tau \approx & \, \mathbb{E}_\tau \sum_{t=\tau}^{\infty} (\theta_W \beta)^{t-\tau} (\widehat{\mathcal{M}}_{W,t} + \widehat{MRS}_t + \pi_{P(\tau,t)} - \gamma_W \pi_{P,(\tau-1,t-1)}) \\ & - \mathbb{E}_\tau \sum_{t=\tau}^{\infty} \frac{\epsilon_W \varphi}{1-h_N} \left(\gamma_W \pi_{P,(\tau-1,t-1)} - \pi_{P,(\tau,t)} - W_t \right) \end{aligned}$$

where we have defined

$$\nu_W = \frac{1 + \epsilon_W \varphi (1-h_N)^{-1}}{1 - \theta_W \beta}$$

In order to apply the general recursive solution from Appendix A, section 7.1.5, we define:

$$\begin{aligned} y_\tau &= \frac{1 + \epsilon_W \varphi (1-h_N)^{-1}}{1-\theta_W \beta} \widehat{\mathcal{W}}_\tau \\ x_t &= \widehat{\mathcal{M}}_{W,t} + \widehat{MRS}_t + \frac{\epsilon_W \varphi}{1-h_N} W_t \\ z_{1,(\tau,t)} &= \pi_{P(\tau,t)} - \gamma_W \pi_{P,(\tau-1,t-1)} \\ z_{2,(\tau,t)} &= -\frac{\epsilon_W \varphi}{(1-h_N)} \left(\gamma_W \pi_{(\tau-1,t-1)} - \pi_{(\tau,t)} \right) \\ z_{\tau,t} &= z_{1,(\tau,t)} - z_{2,(\tau,t)} \end{aligned}$$

It is clear that $z_{\tau,t}$ satisfies the conditions

$$z_{t_1,t_3} = z_{t_1,t_2} + z_{t_2,t_3} \text{ and } z_{t,s} = -z_{s,t}$$

since these conditions are preserved under linear operations. For the sake of clarity we repress the expectation operator in what follows.[30] Inserting the relevant quantities into equation (136) we get

$$\begin{aligned}\nu_W \widehat{\mathcal{W}}_\tau &= \nu_W \widehat{\mathcal{W}}_{\tau+1} + \widehat{\mathcal{M}}_{W,\tau} + \widehat{MRS}_\tau + \frac{\epsilon_W \varphi}{1-h_N} W_\tau \\ &+ \frac{\theta_W \beta}{1-\theta_W \beta} \pi_{P(\tau,\tau+1)} - \gamma_W \pi_{P,(\tau-1,\tau)} \\ &- \frac{\theta_W \beta}{1-\theta_W \beta} \frac{\epsilon_W \varphi}{(1-h_N)} \left(\gamma_W \pi_{P,(\tau-1,\tau)} - \pi_{P,(\tau,\tau+1)}\right)\end{aligned}$$

Rewriting gives

$$\widehat{\mathcal{W}}_\tau = (\theta_w \beta) \widehat{\mathcal{W}}_{\tau+1} + \kappa_1 \left(\widehat{\mathcal{M}}_{W,\tau} + \widehat{MRS}_\tau\right) + \kappa_2 W_\tau \tag{93}$$

$$+ (\theta_W \beta) \left(\pi_{P(\tau,\tau+1)} - \gamma_W \pi_{P,(\tau-1,\tau)}\right) \tag{94}$$

where $\kappa_1 = \frac{1-\theta_W \beta}{\mathfrak{a}}$, $\kappa_2 = \kappa_1 \cdot (\mathfrak{a} - 1)$ and $\mathfrak{a} = 1 + \epsilon_w \varphi (1-h_N)^{-1}$.

We now turn to the law of motion for the wage index given in equation (39). Again similarly to the price index case, we apply Taylor approximation to the law of motion for the wage index and get

$$\widehat{\mathcal{W}}_t = \frac{\theta_W}{1-\theta_W} \left(\pi_{P,(t-1,t)} - \widehat{W}_{t-1} - \gamma_W \pi_{P,(t-2,t-1)}\right) + \frac{1}{1-\theta_W} \widehat{W}_t \tag{95}$$

which we can rewrite as

$$\widehat{\mathcal{W}}_t = \widehat{W}_t + \frac{\theta_W}{1-\theta_W} \left(\pi_{W,(t-1,t)} - \pi_{Z,(t-1,t)} - \gamma_W \pi_{P,(t-2,t-1)}\right) \tag{96}$$

where, as previously defined, $\pi_{W,(t-1,t)} = \widehat{W}_t - \widehat{W}_{t-1} + \pi_{P,(t-1,t)} + \pi_{Z,(t-1,t)}$. To avoid clutter in the following derivation we hereafter write time indices of the type $(t-1,t)$ as simply t. Now inserting equations (95) and (96) into equation (94) gives

$$\begin{aligned}\frac{\theta_W}{1-\theta_W} \left(\pi_{W,t} - \pi_{Z,t} - \gamma_W \pi_{P,t-1}\right) &= (\theta_w \beta) \frac{\theta_W}{1-\theta_W} \left(\pi_{P,t+1} - \widehat{W}_t - \gamma_W \pi_{P,t}\right) \\ &+ \frac{\theta_W \beta}{1-\theta_W} \widehat{W}_{t+1} + \kappa_1 \left(\widehat{\mathcal{M}}_{W,t} + \widehat{MRS}_t\right) \\ &+ (\theta_W \beta) \left(\pi_{P,t+1} - \gamma_W \pi_{P,t}\right) \\ &+ \kappa_2 \widehat{W}_t - \widehat{W}_t\end{aligned}$$

Multiply both sides with $\frac{1-\theta_W}{\theta_W}$, isolate $\pi_{W,t}$, and we get

$$\begin{aligned}\pi_{W,t} &= (\theta_w \beta) \left(\pi_{P,t+1} - \widehat{W}_t - \gamma_W \pi_{P,t}\right) \\ &+ \beta \widehat{W}_{t+1} + \kappa_W \left(\widehat{\mathcal{M}}_{W,t} + \widehat{MRS}_t\right) \\ &+ \kappa_P \left(\pi_{P,t+1} - \gamma_W \pi_{P,t}\right) \\ &+ \kappa_L \widehat{W}_t - \frac{1-\theta_W}{\theta_W} \widehat{W}_t + \pi_{Z,t} + \gamma_W \pi_{P,t-1}\end{aligned}$$

[30] Bear in mind that all quantities are expected values conditional on information available at time τ. Suppressing the expectation operator has no bearing on the calculations since it is preserved over linear operations.

where $\kappa_W = \frac{(1-\theta_W)(1-\theta_W\beta)}{\theta_W \mathfrak{a}}$, $\kappa_L = \kappa_W(\mathfrak{a}-1)$, and $\kappa_P = (1-\theta_W)\beta$. Add and subtract $\beta\pi_{W,t+1}$ from the right side and we get

$$\begin{aligned}
\pi_{W,t} &= \beta\pi_{W,t+1} + \kappa_W\left(\widehat{\mathcal{M}}_{W,t} + \widehat{MRS}_t\right) \\
&\quad + (\theta_w\beta)\left(\pi_{P,t+1} - \widehat{W}_t - \gamma_W\pi_{P,t}\right) \\
&\quad + \beta\widehat{W}_{t+1} + \kappa_L\widehat{W}_t \\
&\quad + \kappa_P\left(\pi_{P,t+1} - \gamma_W\pi_{P,t}\right) \\
&\quad - \frac{1-\theta_W}{\theta_W}\widehat{W}_t + \pi_{Z,t} + \gamma_W\pi_{P,t-1} \\
&\quad - \beta\widehat{W}_{t+1} + \beta\widehat{W}_t - \beta\pi_{Z,t+1} - \beta\pi_{P,t+1}
\end{aligned} \tag{97}$$

where we use, as previously defined, $\pi_{W,t+1} = \widehat{W}_{t+1} - \widehat{W}_t + \pi_{Z,t+1} + \pi_{P,t+1}$. Notice now that collecting coefficients in terms containing the variable $\widehat{W}_t$ gives

$$\begin{aligned}
\kappa_L - \theta_W\beta - \frac{1-\theta_W}{\theta_W} + \beta &= \kappa_W(\mathfrak{a}-1) - \theta_W\beta - \frac{1-\theta_W}{\theta_W} + \beta \\
&= \frac{(1-\theta_W)(1-\theta_W\beta)}{\theta_W} - \theta\beta - \frac{1-\theta_W}{\theta_W} + \beta - \kappa_W \\
&= -\kappa_W
\end{aligned}$$

and we can rewrite equation (97) as

$$\begin{aligned}
\pi_{W,t} &= \beta\pi_{W,t+1} + \kappa_W\Omega_t + (\theta_w\beta)\left(\pi_{P,t+1} - \gamma_W\pi_{P,t}\right) \\
&\quad + \kappa_P\left(\pi_{P,t+1} - \gamma_W\pi_{P,t}\right) + \pi_{Z,t} + \gamma_W\pi_{P,t-1} \\
&\quad - \beta\pi_{Z,t+1} - \beta\pi_{P,t+1}
\end{aligned}$$

where $\Omega_t = \left(\widehat{\mathcal{M}}_{W,t} + \widehat{MRS}_t - \widehat{W}_t\right)$. Now add and subtract $\gamma_W\beta\pi_{P,t}$ and we can write

$$\begin{aligned}
\pi_{W,t} &= \beta\pi_{W,t+1} + \kappa_W\Omega_t + \gamma_W\left(\pi_{P,t-1} - \beta\pi_{P,t}\right) \\
&\quad + (\theta_w\beta)\left(\pi_{P,t+1} - \gamma_W\pi_{P,t}\right) + \kappa_P\left(\pi_{P,t+1} - \gamma_W\pi_{P,t}\right) \\
&\quad + \pi_{Z,t} - \beta\pi_{Z,t+1} - \beta\pi_{P,t+1} + \beta\gamma_W\pi_{P,t}
\end{aligned}$$

Collect like terms to get

$$\begin{aligned}
\pi_{W,t} &= \beta\pi_{W,t+1} + \kappa_W\left(\widehat{\mathcal{M}}_{W,t} + \widehat{MRS}_t - \widehat{W}_t\right) + \gamma_W\left(\pi_{P,t-1} - \beta\pi_{P,t}\right) \\
&\quad + \pi_{Z,t} - \beta\pi_{Z,t+1} \\
&\quad + \pi_{P,t+1}(\theta_W\beta + \kappa_P - \beta) \\
&\quad + \pi_{P,t}\gamma_W(\beta - \theta_W\beta - \kappa_P)
\end{aligned}$$

Recall that all variables at time $t+1$ are expected values at time t. We can thus write

$$\begin{aligned}
\pi_{Z,t} - \beta\mathbb{E}_t[\pi_{Z,t+1}] &= \pi_{Z,t} - \beta\mathbb{E}_t[\rho_Z\pi_{Z,t} + \epsilon_{Z,t}] \\
&= (1-\beta\rho_Z)\pi_{Z,t}
\end{aligned}$$

We then get

$$\pi_{W,t} = \beta\pi_{W,t+1} + \kappa_W\Omega_t + \gamma_W\left(\pi_{P,t-1} - \pi_{P,t}\right) + (1-\beta\rho_Z)\pi_{Z,t} + x_t \tag{98}$$

where

$$x_t = \pi_{P,t+1}(\theta_W\beta + \kappa_P - \beta) - \pi_{P,t}\gamma_W(\theta_W\beta + \kappa_P - \beta)$$

We contend that $x_t = 0$. For this to be true it most hold that

$$(\theta_W\beta + \kappa_P - \beta) = 0$$

But this is clearly so since $\kappa_P = (1-\theta_W)\beta$. The New-Keynesian Phillips curve for wage inflation in the non-suppressed notation is thus given by

$$\begin{aligned}\pi_{W,t} = \beta\mathbb{E}_t[\pi_{W,t+1}] + \kappa_W\left(\mu_{W,t} + \widehat{MRS}_t - \widehat{W}_t\right)\\ + \gamma_W\left(\pi_{P,t-1} - \beta\pi_{P,t}\right) + (1-\beta\rho_Z)\pi_{Z,t}\end{aligned} \tag{99}$$

5.4.3 Import and export prices

Since the derivation of import and export prices are almost identical to those of domestic prices and wages they are omitted.

The New-Keynesian Phillips curve for import prices is given by

$$\pi_{F,t} = \kappa_F(\mu_{F,t} + \widehat{MC}^*_{H,t} - \widehat{P}_{F,t}) + \beta\mathbb{E}_t[\pi_{F,t+1}] + \gamma_H(\pi_{P,t-1} - \beta\pi_{P,t}) \tag{100}$$

where

$$\kappa_F = \frac{(1-\theta_F)(1-\theta_F\beta)}{\theta_F}\frac{1-\psi_F}{1-\psi_F-\psi_F\varepsilon_F}$$

and

$$p_{F,t} - p_{F,t-1} + \pi_{P,t} = \pi_{F,t}$$

The New-Keynesian Phillips curve for export prices is given by

$$\pi^*_{H,t} = \kappa^*_H(\mu^*_{H,t} + \widehat{MC}^*_{H,t} - \widehat{P}^*_{H,t} - \widehat{S}_t) + \beta\mathbb{E}_t[\pi^*_{H,t+1}] + \gamma^*_H(\pi^*_{P,t-1} - \beta\pi^*_{P,t}) \tag{101}$$

where

$$\kappa^*_H = \frac{(1-\theta^*_H)(1-\theta^*_H\beta)}{\theta^*_H}$$

and

$$p^*_{H,t} - p^*_{H,t-1} + \pi^*_{P,t} = \pi^*_{H,t}$$

Note that the difference between κ_F, κ^*_H and κ_H is inconsequential for the estimation process since there is only one degree of freedom, and we thus calibrate all parameters other than θ.[31] Since we model the foreign economy as a simple VAR model on output, inflation and interest rates, we must map those variables to the foreign marginal cost. We do so in an ad-hoc fashion, defining

$$\widehat{MC}^*_{H,t} = \eta^*_{mc,y}\widehat{Y}^*_t \tag{102}$$

[31] This assumes, naturally, that the prior distributions are sensible.

5.4.4 Exchange rates

Recall that we define the real exchange rate as

$$S_t = \xi_t \frac{P_t^*}{P_t}$$

Thus, the gross growth of the real exchange rate is a stationary series and we write

$$\frac{S_t}{S_{t-1}} = \frac{\xi_t}{\xi_{t-1}} \frac{\Pi_{P,t}^*}{\Pi_{P,t}}$$

Percentage deviation from the steady state is given by

$$\widehat{S}_t = \widehat{S}_{t-1} + \pi_{\xi,t} + \pi_{P,t}^* - \pi_{P,t} \tag{103}$$

Terms of trade are defined as

$$T_t = \frac{P_{F,t}}{\xi_t P_{H,t}^*}$$

which in stationary form we write as

$$T_t = \frac{\overline{P}_{F,t}}{\overline{P}_{H,t}^*} \frac{1}{\xi_t} \frac{P_t}{P_t^*} = \frac{1}{S_t} \frac{\overline{P}_{F,t}}{\overline{P}_{H,t}}$$

and log-linearising gives

$$\widehat{T}_t = \widehat{P}_{F,t} - \widehat{P}_{H,t}^* - \widehat{S}_t \tag{104}$$

5.5 Market equilibrium

5.5.1 Export market

Equation (200) specifies the equilibrium production of specialised exports as

$$EX_{E,t} = \min\left(Z_{E,t} K_{E,t}, \frac{\alpha_e}{1-\alpha_E} EX_{g,t}\right)$$

which can be written stationary form as

$$\overline{EX}_{E,t} = \min(Z_{E,t}\mathcal{K}, \frac{\alpha_e}{1-\alpha_E} \overline{EX}_{g,t})$$

We additionally assume that in steady state it holds that:

$$Z_E \mathcal{K} \leq \frac{\alpha_E}{1-\alpha_E} EX_g$$

By definition we have $EX_E = \frac{\alpha_E}{1-\alpha_E} EX_g$. By the argument given in Appendix A, section 7.1.3, regarding linearisation of the minimum operator we can write

$$\widehat{EX}_{E,t} = \widehat{Z}_{E,t} \tag{105}$$

Turning to the generic export firms, we have from equation (199) that

$$EX_{g,t} = (1-\alpha_E)\alpha^* \left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta} \frac{\left(1-\Gamma^*_{H,t} - EX_{g,t}\frac{\partial \Gamma^*_{H,t}}{\partial EX_{g,t}}\right)^{\eta}}{1-\Gamma^*_{H,t}} A^*_t$$

Detrended, the equation takes the form

$$\overline{EX}_{g,t} = (1-\alpha_E)\alpha^* \left(\overline{P}^*_{H,t}\right)^{-\eta} \frac{\left(1-\Gamma^*_{H,t} - EX_{g,t}\frac{\partial \Gamma^*_{H,t}}{\partial EX_{g,t}}\right)^{\eta}}{1-\Gamma^*_{H,t}} \overline{A}^*_t \frac{Z^*_t}{Z_t}$$

In order to log-linearise the equation above we use mixed methods. Applying the natural logarithm function to both sides of the equation we get

$$\begin{aligned}\ln \overline{EX}_{g,t} &= \ln((1-\alpha_E)\alpha^*) - \eta \ln \overline{P}^*_{H,t} \\ &\quad + \eta \ln\left(1-\Gamma^*_{H,t} - EX_{g,t}\frac{\partial \Gamma^*_{H,t}}{\partial EX_{g,t}}\right) - \ln(1-\Gamma^*_{H,t}) \\ &\quad + \ln \overline{A}^*_t + \ln Z^*_t - \ln Z_t\end{aligned}$$

We can use the approximation identity

$$k \ln X_t = k \ln X + k\widehat{X}_t, \quad \text{for } k \in \mathbb{R}$$

to get

$$\begin{aligned}\eta \ln \overline{P}^*_{H,t} &= \eta \ln \overline{P}^*_H + \eta \widehat{P}^*_H \\ \ln \overline{A}^*_t &= \ln \overline{A}^* + \widehat{A}^* \\ \ln \overline{Z}_{D,t} &= \widehat{Z}_{D,t}\end{aligned}$$

where we have defined $Z_{D,t} = \frac{Z_t}{Z^*_t}$. Further, recall the definition of the export adjustment cost, which at equilibrium becomes

$$\Gamma^*_{H,t} = \frac{\theta^*_M}{2}\left(\frac{EX^*_{g,t}}{A^*_t}\frac{A^*_{t-1}}{EX^*_{g,t-1}} - 1\right)^2$$

It is immediately clear that

$$\Gamma^*_{H,t}|_{ss} = \left.\frac{\partial \Gamma^*_{H,t}}{\partial X_t}\right|_{ss} = 0$$

where $X_t \in \{EX^*_{g,t}, EX^*_{g,t-1}, A^*_t, A^*_{t-1}\}$. We thus get

$$\widehat{EX}_{g,t} = \widehat{A}^*_t - \eta \widehat{P}^*_H - \widehat{Z}_{D,t} + g(\widehat{EX}_{g,t}, \widehat{A}^*_t, \widehat{EX}_{g,t-1}, \widehat{A}^*_{t-1}) \tag{106}$$

where $g(\widehat{EX}_{g,t}, \widehat{A}^*_t, \widehat{EX}_{g,t-1}, \widehat{A}^*_{t-1})$ contains the approximation of the two terms

$$\eta \ln\left(1-\Gamma^*_{H,t} - EX_{g,t}\frac{\partial \Gamma^*_{H,t}}{\partial EX_{g,t}}\right), \text{ and } \quad \ln(1-\Gamma^*_{H,t})$$

which we now turn to writing in terms of deviations from steady state. To that end we use Taylor approximation. For the sake of parsimony and brevity we write the approximation in a one-dimensional general form. We start by looking at the latter term. We have

$$\ln(1-\Gamma^*_{H,t}) \approx -\left.\frac{\partial \Gamma^*_{H,t}}{\partial X_t}\right|_{ss}(1-\Gamma^*_{H,t})^{-1}|_{ss}(X_t - X)$$

Since $\left.\frac{\partial \Gamma^*_{H,t}}{\partial X_t}\right|_{ss} = 0$ we get

$$\ln(1-\Gamma^*_{H,t}) \approx 0$$

Turning to the other term, let us begin by defining, for ease of exposition, the expression

$$\mathfrak{G}_{X,t} = 1 - \Gamma^*_{H,t} - EX_{g,t}\frac{\partial \Gamma^*_{H,t}}{\partial X_t}$$

and we have

$$\begin{aligned}
\ln\left(\mathfrak{G}_{EX_g,t}\right) &\approx \ln\left(\mathfrak{G}_{X,t}|_{ss}\right) - \frac{\left.\frac{\partial \Gamma^*_{H,t}}{\partial X_t}\right|_{ss}}{\mathfrak{G}_{EX_g,t}\big|_{ss}}(X_t - X) \\
&\quad - \frac{\left.\left(\frac{\partial EX_t}{\partial X_t}\frac{\partial \Gamma^*_{H,t}}{\partial EX_{g,t}} + EX_{g,t}\frac{\partial^2 \Gamma^*_{H,t}}{\partial X_t \partial EX_{g,t}}\right)\right|_{ss}}{\mathfrak{G}_{EX_g,t}\big|_{ss}}(X_t - X) \\
&= -EX_g \left.\frac{\partial^2 \Gamma^*_{H,t}}{\partial X_t \partial EX_{g,t}}\right|_{ss}(X_t - X)
\end{aligned}$$

Now we need to find the second partial derivative of the export adjustment cost function with respect to the four variables. We have

$$\frac{\partial \Gamma^*_{H,t}}{\partial EX_t} = \theta^*_M \left(\frac{EX_{g,t}}{A^*_t}\frac{A^*_{t-1}}{EX_{g,t-1}} - 1\right)\cdot\left(\frac{1}{A^*_t}\frac{A^*_{t-1}}{EX_{g,t-1}}\right)$$

and we calculate

$$\left.\frac{\partial^2 \Gamma^*_{H,t}}{\partial EX^2_{g,t}}\right|_{ss} = \left.\phi^*_m\left(\frac{1}{A^*_t}\frac{A^*_{t-1}}{EX_{g,t-1}}\right)^2\right|_{ss} = \frac{\phi^*_m}{EX^2_g}$$

$$\left.\frac{\partial^2 \Gamma^*_{H,t}}{\partial EX_{g,t-1}\partial EX_{g,t}}\right|_{ss} = \frac{-\phi^*_m}{EX^2_g}$$

$$\left.\frac{\partial^2 \Gamma^*_{H,t}}{\partial A^*_t \partial EX_{g,t}}\right|_{ss} = \frac{-\phi^*_m}{A^* EX_g}$$

$$\left.\frac{\partial^2 \Gamma^*_{H,t}}{\partial A^*_{t-1}\partial EX_{g,t}}\right|_{ss} = \frac{\phi^*_m}{A^* EX_g}$$

and we can write

$$\eta \ln\left(1-\Gamma^*_{H,t}-EX_{g,t}\frac{\partial\Gamma^*_{H,t}}{\partial EX_{g,t}}\right)-\ln(1-\Gamma^*_{H,t}) = -\eta EX_g\frac{\phi^*_m}{EX_g^2}\left(EX_{g,t}-EX_g\right)$$
$$+\eta EX_g\frac{\phi^*_m}{EX_g^2}\left(EX_{g,t-1}-EX_g\right)$$
$$+\eta EX_g\frac{-\phi^*_m}{A^*EX_g}\left(A^*_t-A\right)$$
$$-\eta EX_g\frac{\phi^*_m}{A^*EX_g}\left(A^*_{t-1}-A\right)$$

which can be rewritten in terms of $g(\cdot)$ as

$$g(\widehat{EX}_{g,t},\widehat{A}^*_t,\widehat{EX}_{g,t-1},\widehat{A}^*_{t-1}) = \eta\phi^*_m\left(\widehat{EX}_{g,t-1}-\widehat{EX}_{g,t}-\widehat{A}_{t-1}+\widehat{A}_t\right) \tag{107}$$

Inserting this expression for $g(\cdot)$ into equation (106) we get

$$\widehat{EX}_{g,t} = \widehat{A}^*_t-\eta\widehat{P}^*_H-\widehat{Z}_{D,t}+\eta\phi^*_m\left(\widehat{EX}_{g,t-1}-\widehat{EX}_{g,t}-\widehat{A}_{t-1}+\widehat{A}_t\right) \tag{108}$$

collecting terms and simplifying we finally get

$$\widehat{EX}_{g,t} = \widehat{A}^*_t-\frac{\eta}{1+\eta\phi^*_m}\widehat{P}^*_H+\frac{\eta\phi^*_m}{1+\eta\phi^*_m}\left(\widehat{EX}_{g,t-1}-\widehat{A}_{t-1}\right)-\frac{1}{1+\eta\phi^*_m}\widehat{Z}_{D,t} \tag{109}$$

Finally, by definition we have $EX_t = EX_{g,t}+EX_{E,t}$. Now, assuming that $\frac{EX_E}{EX}=\alpha_E$, implies $EX_g=(1-\alpha_E)EX$. Thus we get

$$\widehat{EX}_t = (1-\alpha_E)\widehat{EX}_{g,t}+\alpha_E\widehat{EX}_{E,t} \tag{110}$$

5.5.2 Import market

From equation (202) we have that import is given by

$$IM_t = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta}A_t$$

which detrended becomes

$$\overline{IM}_t = \alpha\overline{P}^{-\eta}_{F,t}\overline{A}_t \tag{111}$$

Via our standard methods we get

$$\widehat{IM}_t = \widehat{A}_t-\eta\widehat{P}_{F,t} \tag{112}$$

5.6 Output and net exports

5.6.1 Aggregate output

Now we turn to aggregate domestic output per capita. From equation (201) we have

$$Y_{H,t} = Y^h_{H,t}+EX_t$$

Detrending gets us

$$\overline{Y}_{H,t} = \overline{Y}^h_{H,t} + \overline{EX}_t$$

From our discussion in Appendix A, section 7.1.3, we know that we can write

$$\widehat{Y}_{H,t} = \frac{Y_H^h}{Y_H}\widehat{Y}^h_{H,t} + \frac{EX}{Y_H}\widehat{EX}_t$$

But we know that $\frac{Y_H^h}{Y} = (1-\alpha)$ and $\frac{EX}{Y} = \alpha$ from the derivations in Appendix C. It is thus clear that

$$\frac{Y_H}{Y} = \frac{Y_H^h}{Y} + \frac{EX}{Y} = 1$$

That is to say $Y_H = Y$, and we thus get

$$\widehat{Y}_{H,t} = (1-\alpha)\widehat{Y}^h_{H,t} + \alpha\widehat{EX}_t \tag{113}$$

We have yet to express $Y^h_{H,t}$ in percentage deviation from steady state, to that end we recall that equation (198) states

$$Y^h_{H,t} = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} A_t$$

which detrended becomes

$$\overline{Y}^h_{H,t} = (1-\alpha)\left(\overline{P}_{H,t}\right)^{-\eta}\overline{A}_t$$

Using our standard methods we get

$$\widehat{Y}^h_{H,t} = -\eta\widehat{P}_{H,t} + \widehat{A}_t \tag{114}$$

5.6.2 Real GDP

Real GDP per capita is given by equation (50)

$$Y_t = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{1-\eta} T_t^\alpha A_t + \frac{\mathcal{E}_t P^*_{H,t}}{P_{Y,t}} EX_t$$

Detrended, we have:

$$\overline{Y}_t = (1-\alpha)\left(\overline{P}_{H,t}\right)^{1-\eta} T_t^\alpha \overline{A}_t + \frac{\mathcal{S}_t \overline{P}^*_{H,t}}{\overline{P}_{Y,t}}\overline{EX}_t$$

Using the substitution method we get

$$\begin{aligned}
Y(1+\widehat{Y}_t) &= (1-\alpha)P_H(1+(1-\eta)\widehat{P}_{H,t})T(1+\alpha\widehat{T})A(1+\widehat{A}_t) \\
&\quad + \mathcal{S}(1+\widehat{S}_t)P^*_H(1+\widehat{P}^*_{H,t})P^*_Y(1-\widehat{P}^*_{Y,t})EX(1+\widehat{EX}_t) \\
&= A(1-\alpha)(1+(1-\eta)\widehat{P}_{H,t})(1+\alpha\widehat{T})(1+\widehat{A}_t) \\
&\quad + EX(1+\widehat{S}_t)(1+\widehat{P}^*_{H,t})(1-\widehat{P}^*_{Y,t})(1+\widehat{EX}_t)
\end{aligned}$$

where we used $P_H = P_H^* = S = T = 1$, as shown in Appendix C. Recall our assumption that multiplication of two percentage deviations from steady state are negligible, which yields

$$\begin{aligned} Y + Y\widehat{Y}_t = (1-\alpha)A(1 + (1-\eta)\widehat{P}_{H,t} + \alpha\widehat{T}_t + \widehat{A}_t) \\ + EX(1 + \widehat{P}^*_{H,t} + \widehat{S}_t - \widehat{P}_{Y,t} + \widehat{EX}_t) \end{aligned}$$

Using $Y = (1-\alpha)A + EX$ from Appendix C, and cancelling out common terms gives

$$\begin{aligned} Y\widehat{Y}_t = A\left((1-\alpha)(1-\eta)\widehat{P}_{H,t} + \alpha\widehat{T}_t + (1-\alpha)\widehat{A}_t)\right) \\ + EX\left(\widehat{EX}_t + \widehat{P}^*_{H,t} + \widehat{S}_t - \widehat{P}_{Y,t}\right) \end{aligned}$$

Now dividing by $Y = A$ and using the fact that $EX = \alpha Y$, we get

$$\begin{aligned} \widehat{Y}_t = (1-\alpha)(1-\eta)\widehat{P}_{H,t} + (1-\alpha)\widehat{A}_t \\ + \alpha(\widehat{S}_t + \widehat{P}^*_{H,t} + \widehat{EX}_t) + \alpha((1-\alpha)\widehat{T} - \widehat{P}_{Y,t}) \end{aligned}$$

Recall that we define

$$P_{Y,t} = P_t P_{F,t}^{-\alpha}(\xi_t P^*_{H,t})^{\alpha} = P_t T_t^{-\alpha}$$

and we have

$$\widehat{P}_{Y,t} = -\alpha\widehat{T} \tag{115}$$

We rewrite our expression for real GDP as

$$\widehat{Y}_t = (1-\alpha)(1-\eta)\widehat{P}_{H,t} + (1-\alpha)\widehat{A}_t + \alpha(\widehat{S}_t + \widehat{P}^*_{H,t} + \widehat{EX}_t) - \widehat{P}_{Y,t} \tag{116}$$

We can further rewrite the last equation using the already established relations

$$\widehat{IM}_t = \widehat{A}_t - \eta\widehat{P}_{F,t}, \quad \text{and } (1-\alpha)P_{H,t} = \alpha P_{F,t}$$

along with the definition of the terms of trade and the GDP deflator, to get

$$\widehat{Y}_t = \widehat{A}_t + \widehat{EX}_t - \widehat{IM}_t$$

The output gap can thus be expressed as the weighted percentage deviation of GDP's subcomponents.[32]

5.6.3 Net export

Net export per capita is given by equation (48):

$$NX_t = EX_t - T_t IM_t$$

Detrended it becomes

$$\overline{NX}_t = \overline{EX}_t - T_t\overline{IM}_t$$

[32]Recall that $\widehat{A}_t = \frac{C}{Y}\widehat{C}_t + \frac{I}{Y}\widehat{I}_t + \frac{G}{Y}\widehat{G}_t + \frac{M}{A}\widehat{M}_t$.

Since we can't assume that the steady state of net exports is non-zero, we can't derive NX_t as a percentage deviation from its steady state. We therefore find the deviation from the steady state as a percentage of GDP. First order Taylor approximation gives:

$$\begin{aligned}
\overline{NX}_t - NX &\approx (\overline{EX}_t - EX) - T(\overline{IM}_t - IM) - IM(T_t - T) \\
&= EX\frac{(\overline{EX}_t - EX)}{EX} - T \cdot IM\frac{(\overline{IM}_t - IM)}{IM} - T \cdot IM\frac{(T_t - T)}{T} \\
&= EX\widehat{EX}_t - IM\widehat{IM}_t - IM\widehat{T} \\
&= \alpha A\widehat{EX}_t - \alpha A\widehat{IM}_t - \alpha A\widehat{T} \\
&= \alpha A(\widehat{EX}_t - \widehat{IM}_t - \widehat{T})
\end{aligned}$$

where we again used that $T = 1$, $EX = IM = \alpha A$, justified in Appendix C. Next we divide with the steady state of real GDP and get

$$\widehat{NX}_t = \frac{\overline{NX}_t - NX}{Y} = \widehat{NX}_t = \alpha(\widehat{EX}_t - \widehat{IM}_t - \widehat{T}) \tag{117}$$

where we use the previously stated fact that $Y = A$.

5.7 Public policy

5.7.1 Monetary policy

In stationary form, monetary policy response function is given by:

$$\frac{R_t}{R} = Z_{R,t}\left(\frac{R_{t-1}}{R}\right)^{\xi_R}\left[\left(\frac{\Pi_{P,t}}{\Pi_P}\right)^{\phi_P}\left(\frac{\overline{Y}_t}{Y}\right)^{\phi_Y}\left(\frac{\overline{Y}_t}{\overline{Y}_{t-1}}\right)^{\phi_{\Delta Y}}\right]^{1-\xi_R} \tag{118}$$

By simply taking the logarithm of both sides, we get

$$\widehat{R}_t = \xi_R\widehat{R}_{t-1} + (1-\xi_R)\left[\phi_P\pi_{P,t} + \phi_Y\widehat{Y}_t + \phi_{\Delta Y}\left(\widehat{Y}_t - \widehat{Y}_{t-1}\right)\right] + z_{R,t} \tag{119}$$

where we used the assumption $Z_R = 1$ and the fact that for a stationary series X_t it follows that

$$\ln\left(\frac{X_t}{X_{t-1}}\right) = \ln\left(\frac{X_t X}{X_{t-1}X}\right) = \ln\left(\frac{X_t}{X}\right) - \ln\left(\frac{X_{t-1}}{X}\right) = \widehat{X}_t - \widehat{X}_{t-1}$$

5.7.2 Government spending

Lastly, to close the model we assume that government spending is exogenous, dictated by the following stochastic process

$$\widehat{G}_t = \rho_g\widehat{G}_{t-1} + \epsilon_{G,t} \tag{120}$$

5.8 Exogenous shocks

A stochastic process of the form

$$x_t = \alpha^{1-\rho} x_{t-1}^{\rho} e^{\varepsilon_t}, \quad \rho \in (0,1)$$

has the steady state $x = \alpha$ and we can write

$$\ln(x_t) - \ln(\alpha) = (1-\rho)\ln(\alpha) + \rho \ln(x_{t-1}) + \varepsilon_t - \ln(\alpha)$$

which reduces to

$$\widehat{x}_t = \rho\left(\ln(x_{t-1}) - \ln(\alpha)\right) + \varepsilon_t = \rho \widehat{x}_{t-1} + \varepsilon_t$$

Therefore, all the presented stochastic processes have this form in percentage deviation from the steady state, given by:

$$\widehat{Z}_{k,t} = \rho_k \widehat{Z}_{k,t-1} + \varepsilon_{k,t} \tag{121}$$

$$\mu_{l,t} = \rho_{\mu_l} \mu_{l,t-1} + \varepsilon_{\mu_l,t} \tag{122}$$

$$\mu^*_{H,t} = \rho^*_{\mu_H} \mu^*_{H,t-1} + \varepsilon^*_{\mu_H,t} \tag{123}$$

for $k \in \{B, C, I, D, H, R\}$ and $l \in \{H, F, W\}$. Lastly, the permanent technology shock is written

$$\pi_{Z,t} = \rho \pi_{Z,t-1} + \varepsilon_{Z,t} \tag{124}$$

All shocks are normally distributed with zero mean, and non-identical variances.

5.9 Foreign economy

Since the domestic economy is treated as a small economy, we can assume that the foreign economy is independent of domestic shocks. For our purposes we assume that the foreign economy is sufficiently well described by a simple VAR model. Let $x_t^* = [\widehat{Y}_t^* \quad \widehat{\pi}_t^* \quad \widehat{R}_t^*]'$ and we presume that the foreign economy evolves in accordance with

$$x^*_{t+1} = \Xi x^*_t + \varepsilon^*_{t+1} \tag{125}$$

where $\varepsilon^*_{t+1} \sim N(0, \Sigma^*)$ and Ξ, Σ^* are 3×3 matrices. In estimation, we assume that Σ^* is a diagonal matrix.

6 Summary of the linearised model

Following is a list of the estimated model equations. In front of each equation is the equation number in the handbook. Note that the UIP has been modified with a backward-looking term and a smoothing parameter.

Prices and costs

(60) **:** $\widehat{S}_t = \delta_S E[\widehat{S}_{t+1}] + (1-\delta_S)\widehat{S}_{t-1} - (\widehat{R}_t - E[\pi_{P,t+1}]) + (\widehat{R}^*_t - E[\pi^*_{P,t+1}]) - \gamma_{B,t}$

$$(103): \ \widehat{S}_t = \widehat{S}_{t-1} + \pi_{\xi,t} + \pi^*_{P,t} - \pi_{P,t}$$

$$(104): \ \widehat{T}_t = \widehat{P}_{F,t} - \widehat{P}^*_{H,t} - \widehat{S}_t$$

$$(57): \ \widehat{R}^K_t = \widehat{N}_{S,t} + \widehat{W}_t - \widehat{K}_t$$

$$(58): \ \widehat{MC}_t = (1-\psi_H)\widehat{W}_t + \psi_H \widehat{R}^K_t - (1-\psi_H)\widehat{Z}_{H,t}$$

$$(72): \ \widehat{P}_{H,t} = -\tfrac{\alpha}{1-\alpha}\widehat{P}_{F,t}$$

$$(81): \ \pi_{H,t} = \kappa_H(\mu_t + \widehat{MC}_{H,t} - \widehat{P}_{H,t}) + \beta \mathbb{E}_t[\pi_{H,t+1}] + \gamma_H(\pi_{P,t-1} - \beta\pi_{P,t})$$

$$(73): \ \pi_{H,t} = \widehat{P}_{H,t} - \widehat{P}_{H,t-1} + \pi_{P,t}$$

$$(101): \ \pi^*_{H,t} = \kappa^*_H(\mu^*_{H,t} + \widehat{MC}^*_{H,t} - \widehat{P}^*_{H,t} - \widehat{S}_t) + \beta \mathbb{E}_t[\pi^*_{H,t+1}] + \gamma^*_H(\pi^*_{P,t-1} - \beta\pi^*_{P,t})$$

$$(73): \ \pi^*_{H,t} = \widehat{P}^*_{H,t} - \widehat{P}^*_{H,t-1} + \pi^*_{P,t}$$

$$(100): \ \pi_{F,t} = \kappa_F(\mu_{F,t} + \widehat{MC}^*_{H,t} - \widehat{P}_{F,t}) + \beta \mathbb{E}_t[\pi_{F,t+1}] + \gamma_H(\pi_{P,t-1} - \beta\pi_{P,t})$$

$$(73): \ \pi_{F,t} = \widehat{P}_{F,t} - \widehat{P}_{F,t-1} + \pi_{P,t}$$

$$(115): \ \widehat{P}_{Y,t} = -\alpha \widehat{T}_t$$

Household decisions

$$(59): \ \widehat{C}_t = \tfrac{h_C}{1+h_c}\widehat{C}_{t-1} + \tfrac{1}{1+h_C}\mathbb{E}_t[\widehat{C}_{t+1}] - \tfrac{1-h_C}{1+h_C}\mathbb{E}_t[\widehat{R}_t - \pi_{t+1}] + \tfrac{1}{1+h_C}\mathbb{E}_t[\pi_{Z,t+1}]$$
$$- \tfrac{h_C}{1+h_C}\pi_{Z,t} - \tfrac{1-h_C}{1+h_C}\mathbb{E}_t[\pi_{Z_C,t+1}]$$

$$(68): \ \widehat{Q}_t(j) = -\mathbb{E}_t\left[\widehat{R_t} - \pi_{P,t+1}\right] + (1-\omega_q)\mathbb{E}_t\left[\widehat{R}^K_{t+1}\right] + \omega_q \mathbb{E}_t\left[\widehat{Q}_{t+1}\right]$$

$$(64): \ \widehat{K}_t = \widehat{U}_t + \widehat{K}_{S,t} - \widehat{\Pi}_{Z,t}$$

$$(66): \ \widehat{K}_{S,t+1} = (1-\delta)(\widehat{K}_{S,t} - \widehat{\Pi}_{Z,t}) + \delta(\widehat{I}_t + \widehat{Z}_{I,t})$$

$$(63): \ \widehat{R}^K_t = \lambda_U \widehat{U}_t$$

$$(71): \ \widehat{I}_t = \tfrac{1}{1+\beta}\left(\beta\mathbb{E}[\widehat{I}_{t+1} + \pi_{Z,t+1}] + \widehat{I}_{t-1} - \pi_{Z,t} + \lambda_I(\widehat{Z}_{I,t} + \widehat{Q}_t)\right)$$

$$(99): \ \pi_{W,t} = \beta\mathbb{E}_t[\pi_{W,t+1}] + \kappa_W(\widehat{MRS}_t - \widehat{W}_t + \mu_{W,t}) + \gamma_W(\pi_{P,t-1} - \beta\pi_{P,t})$$
$$+ (1-\beta\rho)\pi_{Z,t}$$

$$(74): \ \pi_{W,t} = \widehat{W}_t - \widehat{W}_{t-1} + \pi_{P,t} + \pi_{Z,t}$$

$$(91): \ \widehat{MRS}_t = \tfrac{\varphi}{1-h_N}(\widehat{N}_t - h_N\widehat{N}_{t-1}) - \widehat{Z}_{C,t} + \tfrac{1}{1-h_C}(\widehat{C}_t - h_C\widehat{C}_{t-1}) + \tfrac{h_C}{1-h_C}\pi_{Z,t}$$

$$(61): \ \gamma_{B,t} = \theta_B b^*_{H,t+1} + z_{B,t}$$

$$(62): \ b^*_{H,t+1} = \beta^{-1} b^*_{H,t} + \widehat{NX}_t$$

$$(56): \ \widehat{N}_{S,t} = \tfrac{1}{1-\bar{h}}\widehat{N}_t$$

Monetary policy and government

$$(119): \widehat{R}_t = \xi_R \widehat{R}_{t-1} + (1-\xi_R)\left[\phi_P \pi_{P,t} + \phi_Y \widehat{Y}_t + \phi_{\Delta Y}\left(\widehat{Y}_t - \widehat{Y}_{t-1}\right)\right] + z_{R,t}$$

$$(120): \widehat{G}_t = \rho_g \widehat{G}_{t-1} + \epsilon_{G,t}$$

Market equilibrium

$$(114): \widehat{Y}^h_{H,t} = -\eta \widehat{P}_{H,t} + \widehat{A}_t$$

$$(113): \widehat{Y}_{H,t} = (1-\alpha)\widehat{Y}^h_{H,t} + \alpha \widehat{EX}_t$$

$$(55): \widehat{Y}_{H,t} = \psi_H \widehat{K}_t + (1-\psi_H)(\widehat{N}_{S,t} + \widehat{Z}_{H,t})$$

$$(116): \widehat{Y}_t = (1-\alpha)(1-\eta)\widehat{P}_{H,t} + (1-\alpha)\widehat{A}_t + \alpha(\widehat{S}_t + \widehat{P}^*_{H,t} + \widehat{EX}_t) - \widehat{P}_{Y,t}$$

$$(109): \widehat{EX}_{g,t} = \widehat{A}^*_t + \tfrac{\eta\phi^*_m}{1+\eta\phi^*_m}\left(\widehat{EX}_{g,t-1} - \widehat{A}^*_{t-1}\right) - \tfrac{\eta}{1+\eta\phi^*_m}\widehat{P}^*_{H,t} - \tfrac{1}{1+\eta\phi^*_m}\widehat{Z}_{D,t}$$

$$(105): \widehat{EX}_{E,t} = \widehat{Z}_{E,t}$$

$$(110): \widehat{EX}_t = (1-\alpha_E)EX_{E,t} + \alpha_E EX_{g,t}$$

$$(112): \widehat{IM}_t = \widehat{A}_t - \eta \widehat{P}_{F,t}$$

$$(117): \widehat{NX}_t = \alpha(\widehat{EX}_t - \widehat{IM}_t - \widehat{T})$$

where composite parameters are defined in the following way

$$\begin{aligned}
\kappa_H &= \frac{(1-\theta_H)(1-\theta_H\beta)}{\theta_H} \\
\kappa_F &= \frac{(1-\theta_F)(1-\theta_F\beta)}{\theta_F}\frac{1-\psi_F}{1-\psi_F-\psi_F\varepsilon_F} \\
\kappa_W &= \frac{(1-\theta_W)(1-\theta_W\beta)}{\theta_W}\frac{1}{(1+\epsilon_W\varphi(1-h_N)^{-1})} \\
\kappa^*_H &= \frac{(1-\theta^*_H)(1-\theta^*_H\beta)}{\theta^*_H} \\
\omega_q &= (1-\delta)\beta
\end{aligned}$$

Foreign economy

$$(125): x^*_{t+1} = \Xi x^*_t + \varepsilon^*_{t+1}$$

$$(102): \widehat{MC}^*_t = \eta^*_{mc,y}\widehat{A}^*$$

Stochastic processes

(121) **:** $\widehat{Z}_{i,t} = \rho_i \widehat{Z}_{i,t-1} + \varepsilon_{i,t}$

(122) **:** $\mu_{j,t} = \rho_{\mu_j} \mu_{j,t-1} + \varepsilon_{\mu_j,t}$

(123) **:** $\mu^*_{H,t} = \rho^*_{\mu_H} \mu^*_{H,t-1} + \varepsilon^*_{\mu_H,t}$

(124) **:** $\pi_{Z,t} = \rho \pi_{Z,t-1} + \varepsilon_{Z,t}$

where $i \in \{B, C, I, D, H, R\}$ and $j \in \{H, F, W\}$

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7 Appendix A

7.1 Solution methods

We assume that the reader is well versed in optimisation techniques taught in most undergraduate economics programs. In this subsection we give a brief description of less familiar solution methods used to solve the optimisation problems of agents and firms in the upcoming sections.

7.1.1 Lagrangian multiplier on Banach Spaces

We will denote an arbitrary sequence $\{x\}_{n=1}^{\infty}$ as x, i.e. omitting the index. Let $\ell^{\infty} \subset \mathbb{R}^{\mathbb{N}}$ be the space of all bounded sequences over the real numbers and define $\mathcal{X}_k = \prod_1^k \ell^{\infty}$, then an arbitrary element of $\mathcal{X}_k$ is denoted as[33]

$$z = (x_1, x_2, \dots, x_k)$$

where $x_i \in \ell^{\infty}$, for all i. We denote the t-th realisation of z in terms of the t-th realisation of each subsequence in the natural way

$$z_t = (x_{1,t}, x_{2,t}, \dots, x_{k,t})$$

We proceed by defining the functions $f : \mathcal{X}_k \to \mathbb{R}$ and $g : \mathcal{X}_k \to \mathbb{R}^{\mathbb{N}}$ by:

$$f(z) = \mathbb{E}_{\tau}\left[\sum_{t=\tau}^{\infty} h(z_t)\right], \quad h : \mathbb{R}^k \to \mathbb{R}$$

and

$$g(z) = (\zeta_0, \zeta_1, \dots, \zeta_t, \dots)$$

where ζ_t represents the relevant constraints at time t. We assume that h and ζ_t are continuously differentiable functions for all t. Given a sequence $x_i \in \ell^{\infty}$, we can define the norm

$$\|x_i\|_{\infty} = \sup_t |x_{i,t}|$$

and ℓ^{∞} is a Banach space with respect to that norm. A finite product of Banach spaces is itself Banach via the norm

$$\|z\| = \sum_{i=1}^{k} \|x_i\|$$

Thus it is clear that $\mathcal{X}_k$ is a Banach space and the extended methods of Lagrange multipliers applies to the problem of maximising $f(z)$ with the constrain that $g(z) = 0$ (see Zeidler (1995, p. 270-271) for proof and details). Thus there exists $\lambda : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, such that

$$Df(z) = \lambda \circ Dg(z)$$

In particular, we have

$$Df(z) = \langle \lambda, Dg(z) \rangle$$

[33]One can think of $\mathcal{X}_k$ as $\mathbb{R}^{\mathbb{N}}$, by thinking of the k sequences as one sequence interlaced. For example, if $\{v\}_{t=1}^{\infty}$ and $\{u\}_{t=1}^{\infty}$ are two arbitrary sequences, we can map them to the sequence $\{x\}_{t=0}^{\infty}$, where $x_{2k+1} = u_{k+1}$, $x_{2k} = v_{k+1}$.

where $Df(z) : \mathcal{X}_k \to \mathbb{R}$ and $Dg(z) : \mathcal{X}_k \to \mathbb{R}^{\mathbb{N}}$ represent the Fréchet derivatives of f and g, respectively, at z. It is well known that the Fréchet derivatives can be characterised by the directional derivative, i.e.

$$Df(z)e_i = \frac{d}{dt}\left[f(z + te_i)\right]|_{t=0}$$

Thus, a necessary condition for maximisation is given in terms of the partial derivatives of f and g with respect to each variable at each time.[34]

7.1.2 Functional derivatives

Let $J[f]$ be a functional on the normed space of k continuously differentiable functions, $\mathcal{C}_k$, and we say that $J[f]$ is differentiable and $\delta J[h]$ is the principle linear part of J at f if, for $f, h \in \mathcal{C}_k$, we can write

$$J[f + h] - J[f] = \delta J[h] + \varepsilon ||h||$$

such that $\epsilon \to 0$, when $||h|| \to 0$. We have that $J[f]$ attains a local extrema at $\widehat{f}$ only if $\delta J[h] = 0$, for $f = \widehat{f}$ and all admissible functions h (see Gelfand and Fomin (2000)). Using Taylor expansion we can extract the principle linear part as

$$\delta J[h] = \frac{d}{d\varepsilon} J[f + \varepsilon\eta]\Big|_{\varepsilon=0}$$

Note that this is analogous to the finite dimensional case. For a sufficiently well behaved function $f : \mathbb{R}^n \to \mathbb{R}$, we define

$$\frac{d}{d\varepsilon} f[x + \varepsilon h]\Big|_{\varepsilon=0} = \nabla f \cdot h = \langle \nabla f, h \rangle$$

Defining the inner product on $\mathcal{C}_k$ as

$$\langle u, v \rangle = \int_{[a,b]} u(x)v(x)$$

for $u, v \in \mathcal{C}_k$. We get

$$\delta J[f] = \left\langle \frac{\partial J[f]}{\partial f}, \eta \right\rangle$$

for an arbitrary $\eta \in \mathcal{C}_k$, and we call $\frac{\partial J[f]}{\partial f}$ the functional derivative. One of the themes of optimisation theory of functionals is that a vanishing functional derivative is a necessary criterion for a local extrema.

Let us suppose that we are tasked with minimising a functional of the form

$$J[f] = \mathcal{L} = \beta G[f]^{\alpha} - \nu \left(\int_{[a,b]} wf dx - \Theta \right) dx$$

where G is functional, $\nu, \Theta, a, b, \alpha, \beta \in \mathbb{R}$, and $w \in \mathcal{C}_k$. We get

$$\delta J[h] = \beta\alpha G[f]^{\alpha-1} \cdot \delta G[h] - \nu \int_{[a,b]} w\eta dx = 0$$

[34] The details of the interplay between expectations and derivative is omitted here. We can interchange the order of expectations and derivatives by invoking the Leibnitz rule and assuming that the underlying probability distributions are well behaved.

Further suppose that $G[f] = \int_{[a,b]} f^\gamma dx$. Then

$$\delta G[f] = \int_{[a,b]} \gamma f^{\gamma-1} \eta dx$$

and we get

$$\int_{[a,b]} \left(\beta\alpha G[f]^{\alpha-1}\gamma f^{\gamma-1} - \nu w\right) \eta dx = 0$$

Since η is arbitrary, we have from the fundamental lemma of calculus of variations that

$$\beta\alpha G[f]^{\alpha-1}\gamma f^{\gamma-1} - \nu w = 0$$

If we assume $\alpha = \gamma^{-1}$ and $f > 0$, we finally get that J attains a local extrema at f, which satisfies

$$\beta G[f]^{\frac{1-\gamma}{\gamma}} f^{\gamma-1} - \nu w = 0 \tag{126}$$

A result that will become more familiar in following sections. For details and more information see Gelfand and Fomin (2000), Zeidler (1995), Giaquinta and Hildebrandt (2004).

7.1.3 Linearisation methods

Standard general methods

Let y be the steady state of a stationary variable of interest, y_t, then the percentage deviation from steady state, $\widehat{y}_t$, is given by

$$\widehat{y}_t = \frac{y_t - y}{y}$$

A common approximation is

$$\widehat{y}_t \approx \ln\left(\frac{y_t}{y}\right) = \ln y_t - \ln y$$

which is justified by noting that the first order Taylor approximation around 0 gives

$$\ln(1+x) \approx x$$

for a general variable x. Thus we have

$$\widehat{y}_t \approx \ln(1+\widehat{y}_t) = \ln\left(1 + \frac{y_t - y}{y}\right) = \ln\left(\frac{y_t}{y}\right)$$

which is a good approximation when y_t is sufficiently close to its steady state. We illustrate how to find the percentage deviation from steady state for four types of functional forms, which gives us the tools to find the steady state deviations of most encountered cases. We refer to these methods as the standard general methods in the text. The examples are

(1) $y_t = kx_t$, **(2)** $y_t = x_t^k$

(3) $y_t = x_{1,t} \cdot x_{2,t}$, **(4)** $y_t = x_{1,t} + x_{2,t}$

1. The equation $y_t = kx_t$ gives steady state $y = kx$ and we can write

$$\widehat{y}_t = \frac{kx_t}{kx} - 1 = \widehat{x}_t$$

2. The log-lin approximation gives

$$\begin{aligned}\widehat{y}_t &= \ln y_t - \ln y \\ &= k \ln x_t - k \ln x \\ &= k\widehat{x}_t\end{aligned}$$

3. The log-lin approximation gives

$$\begin{aligned}\widehat{y}_t &= \ln y_t - \ln y \\ &= \ln x_{1,t} + \ln x_{2,t} - (\ln x_1 + \ln x_2) \\ &= \widehat{x}_{1,t} + \widehat{x}_{2,t}\end{aligned}$$

4. From the definition $\frac{y_t}{y} - 1 = \widehat{y}_t$, we can equivalently write $y_t \approx y\,(1 + \widehat{y}_t)$. This gives

$$y\,(1 + \widehat{y}_t) \approx x_1\,(1 + \widehat{x}_{1,t}) + x_2\,(1 + \widehat{x}_{2,t})$$

Clearly $y = x_1 + x_2$, which cancel out, and we get

$$\widehat{y}_t = \frac{x_1}{y}\widehat{x}_{1,t} + \frac{x_2}{y}\widehat{x}_{2,t}$$

Substitution method

Let x_t and y_t be stationary time series. The definition of the percentage deviation from steady state allows us to write

$$x_t = (1 + \widehat{x})x, \quad \text{and} \quad y_t = (1 + \widehat{y})y$$

We will presuppose that $\widehat{x}_t^2 \approx \widehat{y}_t^2 \approx \widehat{x}_t\widehat{y}_t \approx 0$. Let $n, m \in \mathbb{N}$, it follows that

$$\begin{aligned}\frac{x_t^n y_t^m}{x^n y^m} &= (1 + \widehat{x})^n (1 + \widehat{y})^m \\ &= \left(\sum_{i=0}^{n} a_i \widehat{x}_t^i\right)\left(\sum_{i=0}^{m} b_i \widehat{y}_t^i\right)\end{aligned}$$

where $a_i = \binom{n}{i}$ and $b_i = \binom{m}{i}$, and thus $a_1 = n$ and $b_1 = m$. The assumption $\widehat{x}^2 \approx \widehat{y}^2 \approx 0$ allows us to write

$$\left(\sum_{i=0}^{n} a_i \widehat{x}_t^i\right)\left(\sum_{i=0}^{m} b_i \widehat{y}_t^i\right) \approx (1 + n\widehat{x}_t)(1 + m\widehat{y}_t)$$

and our presupposition of $\widehat{x}_t\widehat{y}_t \approx 0$ gives

$$(1 + n\widehat{x}_t)(1 + m\widehat{y}_t) \approx 1 + n\widehat{x}_t + m\widehat{y}_t$$

which in turn implies

$$\widehat{x_t^n y_t^m} = \frac{x_t^n y_t^m}{x^n y^m} - 1 \approx n\widehat{x}_t + m\widehat{y}_t$$

The result above can naturally be extended to $n, m \in \mathbb{Z}$. We refer to the method descried above as the substitution method.

Taylor approximation

Let a stationary variable of interest be given by[35]

$$y_t = g(x_t, z_t)$$

We can get the deviation of y_t from its steady state by employing first order Taylor approximation on the natural logarithm of y_t. We get

$$\ln y_t \approx \ln y + \frac{\partial g(x_t, z_t)}{\partial x_t}\bigg|_{ss} g(x,z)^{-1}(x_t - x) + \frac{\partial g(x_t, z_t)}{\partial z_t}\bigg|_{ss} g(x,z)^{-1}(z_t - z)$$

using $\widehat{y_t} \approx \ln y_t - \ln y$ we get

$$\widehat{y_t} \approx \frac{x}{y}\frac{\partial g(x_t, z_t)}{\partial x_t}\bigg|_{ss}\widehat{x}_t + \frac{z}{y}\frac{\partial g(x_t, z_t)}{\partial z_t}\bigg|_{ss}\widehat{z}_t$$

where we have substituted $y^{-1} = g(x,z)^{-1}$. This method is the Taylor approximation method.

Linearisation of the max and min operators

To log-linearise the maximum or minimum operators we can approximate the quantities with the LogSumExp function.[36] We then proceed by finding the Taylor approximation of LogSumExp valuation around the steady state. We will only treat the maximum but remind the reader of the fact that

$$\min(x, y) = -\max(-x, -y)$$

We define the LogSumExp function as

$$\mathfrak{L}(x, z) = \ln(e^x + e^z) \tag{127}$$

Let $y_t = \max(x_t, z_t) \approx \mathfrak{L}(x_t, z_t)$ and the Taylor approximation around the steady state gives

$$y_t \approx y + \frac{e^x}{e^y}(x_t - x) + \frac{e^z}{e^y}(z_t - z) \tag{128}$$

which implies

$$\widehat{y_t} \approx \frac{e^x}{e^y}\frac{x}{y}\widehat{x}_t + \frac{e^z}{e^y}\frac{z}{y}\widehat{z}_t \tag{129}$$

Note that the steady state of y is either x or z. Assume without loss of generality that $y = x$ and we get

$$\widehat{y_t} = \widehat{x}_t + \delta\widehat{z}_t \tag{130}$$

where $\delta = \frac{e^z}{e^y}\frac{z}{y}$. Note that if $y >> z$, then δ is approximately 0, i.e. $\widehat{y_t} \approx \widehat{x}_t$. For the minimum we get a similar result with the roles reversed. Further note that if $x_t \approx z_t$, we get

$$y_t \approx \mathfrak{L}(x_t, x_t) = \ln(e^{x_t} + e^{x_t}) = ln(2) + x_t$$

and we deduce that

$$\widehat{y_t} \approx \widehat{x}_t$$

[35]We demonstrate the two variable case, but the method can easily be extended to the case of arbitrary finite number of variables.

[36]The LogSumExp is a good approximation if we expect one of the variables to be a good deal larger than the other.

7.1.4 Linearisation and the expectation operator

When we encounter the expectation operator we need to be careful in applying the standard log-linearisation methods aforementioned. The biggest reason being Jensen's inequality, which states that for a random variable, X, and a concave function, f, we have the inequality

$$f\left(\mathbb{E}[X]\right) \geq \mathbb{E}\left[f(X)\right]$$

In the case when f is linear the condition holds with equality. Since the natural logarithm operator is a concave function we can't apply our logarithmic approximation directly. Rather, we apply the expectation operator after linearisation. Now since the expectations operator is distributive in the sense that for two random variables X and Y, and a real number, a, we have

$$\mathbb{E}[aX + Y] = a\mathbb{E}[X] + \mathbb{E}[Y]$$

we can safely solve for our variable of interest. Applying the substitution method, it clearly follows that

$$\mathbb{E}[\widehat{xy}_t] \approx \mathbb{E}[\widehat{x}_t] + \mathbb{E}[\widehat{y}_t]$$

More generally, for a function $f(x_t, y_t)$,[37] the first order Taylor-approximation around the steady state (x, y) is given by

$$f(x_t, y_t) \approx f(x, y) + \left.\frac{\partial f(x_t, y_t)}{\partial x_t}\right|_{(x,y)} (x_t - x) + \left.\frac{\partial f(x_t, y_t)}{\partial y_t}\right|_{(x,y)} (y_t - y)$$

Taking expectations conditional on information at time $t-1$ we get

$$\mathbb{E}_{t-1}[f(x_t, y_t)] - f(x, y) \approx \left.\frac{\partial f(x_t, y_t)}{\partial x_t}\right|_{(x,y)} \mathbb{E}_{t-1}[x_t - x] + \left.\frac{\partial f(x_t, y_t)}{\partial y_t}\right|_{(x,y)} \mathbb{E}_{t-1}[y_t - y]$$

Define $z_t = f(x_t, y_t)$ and we can write

$$\mathbb{E}_{t-1}[\widehat{z}_t] \approx \left.\frac{\partial f(x_t, y_t)}{\partial x_t}\right|_{(x,y)} \frac{x}{z}\mathbb{E}_{t-1}[\widehat{x}_t] + \left.\frac{\partial f(x_t, y_t)}{\partial y_t}\right|_{(x,y)} \frac{y}{z}\mathbb{E}_{t-1}[\widehat{y}_t]$$

7.1.5 General recursive solution

Assume that a times series y_t can be written in the following form:

$$y_\tau = \mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \omega^{t-\tau}(x_t + z_{\tau,t})\right]$$

Now we define

$$\varsigma(\tau, t) = \omega^{t-\tau}(x_t + z_{\tau,t})$$

and we can write

$$\varsigma(\tau, t) = \omega\varsigma(\tau + 1, t) + \omega^{t-\tau}(z_{\tau,t} - z_{\tau+1,t})$$

[37] The more general case of a function of finite number of variables has a natural extension.

Thus we have

$$y_\tau = \mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \omega^{t-\tau}(x_t + z_{\tau,t}) \right] \tag{131}$$

$$= \mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \varsigma(\tau, t) \right] \tag{132}$$

$$= x_\tau + z_{\tau,\tau} + \mathbb{E}_\tau \left[\sum_{t=\tau+1}^{\infty} \varsigma(\tau, t) \right] \tag{133}$$

$$= x_\tau + z_{\tau,\tau} + \mathbb{E}_\tau \left[\sum_{t=\tau+1}^{\infty} \omega\varsigma(\tau+1, t) + \omega^{t-\tau}(z_{\tau,t} - z_{\tau+1,t}) \right] \tag{134}$$

We have that

$$\mathbb{E}_\tau \left[\sum_{t=\tau+1}^{\infty} \omega\varsigma(\tau+1, t) \right] = \mathbb{E}_{\tau+1} \left[\sum_{t=\tau+1}^{\infty} \omega\varsigma(\tau+1, t) \right]$$

whence we conclude

$$y_\tau = x_\tau + z_{\tau,\tau} + \omega y_{\tau+1} + \mathbb{E}_\tau \left[\sum_{t=\tau+1}^{\infty} \omega^{t-\tau}(z_{\tau,t} - z_{\tau+1,t}) \right] \tag{135}$$

If we assume that

$$z_{t_1,t_3} = z_{t_1,t_2} + z_{t_2,t_3} \text{ and } z_{t,s} = -z_{s,t}$$

we get that

$$z_{\tau,t} - z_{\tau+1,t} = z_{\tau,t} + z_{t,\tau+1} = z_{\tau,\tau+1}$$

This, in turn, gives us finally

$$y_\tau = x_\tau + \omega\mathbb{E}_\tau \left[y_{\tau+1}\right] + \frac{\omega}{1-\omega}\mathbb{E}_\tau \left[z_{\tau,\tau+1}\right] \tag{136}$$

7.2 Household consumption and investment decisions

It is well known that there exists a solution to the household's decision problem as it is stated in this handbook (See Stokey et al. (1989)). This is generally proven using the Bellman equation and Banach's fixed point theorem. Sufficiency is thus ensured. An extension of the method of Lagrange multipliers to the case of functions between Banach spaces can be used to find a necessary condition and a solution to the households' optimisation problem. We can apply the methods discussed in Appendix A, section 7.1.1, *mutatis mutandis*. Using the notation from the aforementioned section, we set $z = (C, N, B_H, B_H^*, I, U, K_S)$, and define $f : \mathcal{X}_7 \to \mathbb{R}$ and $g : \mathcal{X}_7 \to \mathbb{R}^{\mathbb{N}}$ by:

$$f(z) = \mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \mathcal{U}_t(C_t(j), C_{t-1}, N_t(j)) \right]$$

$$g(z) = ((\zeta_b(z_1, z_0), \zeta_c(z_2, z_1)), \ldots, (\zeta_b(z_t, z_{t-1}), \zeta_c(z_{t+1}, z_t)), \ldots)$$

where ζ_b is identified with the budget constraint:

$$\begin{aligned}\zeta_{b,t} =& P_t\left(C_t + I_t + \Gamma_U(U_t)K_{S,t}\right) + \xi_t B^*_{H,t+1} + E\left[\Lambda_{t,t+1}B_{H,t+1}\right] + TA_t \\ &- \left(R^*_{t-1}(1-\Gamma_{B,t-1})\xi_t B^*_{H,t} + B_{H,t} + W_tN_t + R^K_t U_t K_{S,t} + D_t\right)\end{aligned} \tag{137}$$

and ζ_c represents the constraint on the law of motion for capital:

$$\zeta_{c,t} = K_{S,t+1} - (1-\delta)K_{S,t} + Z_{I,t}I_t\left(1 - \Gamma_{I,t}\left(\frac{I_t}{I_{t-1}}\right)\right)$$

where we have suppressed the reference to household j. In Appendix A, section 7.1.1, we argued that a necessary condition for maximisation can be given in terms of the partial derivatives of f and g with respect to each variable at each time. We thus get first order conditions

$$\mathbb{E}_\tau\left[\sum_{t=\tau}^{\infty}\beta^{t-\tau}\frac{\delta\mathcal{U}_t(C_t(j), C_{t-1}, N_t(j))}{\delta x_t} - \sum_{t=\tau}^{\infty}\left\langle(\lambda_{b,t},\lambda_{c,t}), \frac{\delta g(z_t)}{\delta x_t}\right\rangle\right] = 0 \tag{138}$$

where x_t is an arbitrary control variable at time t. It is readily verified that we can work in the usual context of the Lagrange multipliers. For convenience we define the Lagrangian as

$$\mathcal{L} = \mathbb{E}\left[\sum_{t=\tau}^{\infty}\beta^{t-\tau}\mathcal{U}_t(C_t(j), C_{t-1}, N_t(j)) + \sum_{t=\tau}^{\infty}\lambda_{b,t}\xi_{b,t}(z_t) + \sum_{t=\tau}^{\infty}\lambda_{c,t}\xi_c(z_t)\right]$$

The first order condition with respect to consumption is

$$\lambda_{b,t+i} = -\beta^i\mathbb{E}_t[MU_{C_{t+i}}(j)P^{-1}_{t+i}] \tag{139}$$

It is clear that the budget constraint binds in optimum, i.e. $\lambda_{b,t} \neq 0$, and we can thus write the first order condition with respect to domestic bond holdings as

$$1 = \mathbb{E}_\tau[\Lambda_{t,t+1}]\cdot\frac{\lambda_{b,t}}{\lambda_{b,t+1}} \tag{140}$$

which together imply

$$1 = \mathbb{E}_t\left[\Lambda_{t,t+1}\cdot\beta^{-1}\frac{MU_{C_t}}{MU_{C_{t+1}}}\frac{P_{t+1}}{P_t}\right] \tag{141}$$

For equation (141) to hold independently of the underlying probability density function we must have

$$\Lambda_{t,t+1} = \beta\frac{MU_{C_{t+1}}}{MU_{C_t}}\frac{P_t}{P_{t+1}} \tag{142}$$

Define $R_t = \mathbb{E}_t[\Lambda_{t,t+1}]^{-1}$ and we get

$$1 = R_tE[\Lambda_{t,t+1}] \tag{143}$$

Further, using the first order condition with respect to labour supply, given by

$$\lambda_{b,t+i} = \beta^i \mathbb{E}_t[MU_{N_{t+i}}(j) W_{t+i}^{-1}] \tag{144}$$

in combination with the first order condition with respect to consumption from equation (139), we get the familiar equation

$$1 = -\mathbb{E}_t \left[\frac{MU_{N_t}}{MU_{C_t}} \frac{P_t}{W_t} \right]$$

The FOC with respect to foreign bonds gives

$$1 = \frac{\lambda_{b,t+1}}{\lambda_{b,t}} \mathbb{E}_t \left[\frac{\xi_{t+1}}{\xi_t} R_t^* \left(1 - \Gamma_{B,t}\right) \right] \tag{145}$$

Again using equation (140), we get

$$1 = R_t^* (1 - \Gamma_{B,t}) \mathbb{E}_t \left[\Lambda_{t,t+1} \frac{\xi_{t+1}}{\xi_t} \right] \tag{146}$$

Turning to capital utilisation, the first order condition with respect to U_t gives

$$\frac{R_t^K}{P_t} = \Gamma'(U_t) \tag{147}$$

The first order condition w.r.t. $K_{S,t+1}$ is given by

$$0 = \mathbb{E}_t \left[\lambda_{b,t+1} (\Gamma_{U,t+1} P_{t+1} - R_{t+1}^K U_{t+1}) - \lambda_{c,t+1} (1 - \delta) \right] + \lambda_{c,t} \tag{148}$$

and w.r.t. investment

$$\begin{aligned} 0 = & \lambda_{b,t} P_t - \lambda_{c,t} Z_{I,t} \left(1 - \Gamma_{I,t}(\cdot) - \Gamma'_{I,t}(\cdot) \left(\frac{I_t}{I_{t-1}} \right) \right) \\ & - \mathbb{E}_t \left[\lambda_{c,t+1} Z_{I,t+1} \Gamma'_{I,t+1}(\cdot) \left(\frac{I_{t+1}}{I_t} \right)^2 \right] \end{aligned} \tag{149}$$

We define the real marginal Tobin's Q as $Q_t = \frac{\lambda_{c,t}}{\lambda_{b,t}} P_t^{-1}$.[38] Then equation (148) becomes

$$0 = \mathbb{E}_t \left[\Lambda_{t,t+1} (\Gamma_{U,t+1} P_{t+1} - R_{t+1}^K U_{t+1}) - Q_{t+1} P_{t+1} \Lambda_{t,t+1} (1 - \delta) \right] + Q_t P_t$$

where we have used that $\frac{\lambda_{b,t+1}}{\lambda_{b,t}} = E_t[\Lambda_{t,t+1}]$. We can equivalently write

$$Q_t = \mathbb{E}_t \left[\Lambda_{t,t+1} \frac{P_{t+1}}{P_t} \left(\frac{R_{t+1}^K}{P_{t+1}} U_{t+1} - \Gamma_{U,t+1} + (1 - \delta) Q_{t+1} \right) \right] \tag{150}$$

In terms of Tobin's Q, equation (149) can be written as

$$\begin{aligned} 1 = & Q_t Z_{I,t} \left(1 - \Gamma_{I,t}(\cdot) - \Gamma'_{I,t}(\cdot) \left(\frac{I_t}{I_{t-1}} \right) \right) \\ & + \mathbb{E}_t \left[Q_{t+1} \frac{P_{t+1}}{P_t} \Lambda_{t,t+1} Z_{I,t+1} \Gamma'_{I,t+1}(\cdot) \left(\frac{I_{t+1}}{I_t} \right)^2 \right] \end{aligned} \tag{151}$$

[38] Tobin's Q is generally defined as the value of a unit of installed capital in terms of its replacement cost. Thus at optimum, the marginal Tobin's Q must be such that a unit increment in the real value of the investment is equal to the increment in real cost of the investment. We define real marginal Tobin's Q as the shadow price of physical capital investment relative to the shadow price of income in consumption units.

7.3 Factor markets and marginal cost

7.3.1 Labour supply

A generic domestic firm i wants to maximise the labour services employed at time t, for a given expenditure on labour since their production function is increasing in labour. We have the Lagrangian

$$\mathcal{L} = n^{\frac{\rho-1}{\rho}} \left(\int N_t^{\rho}(i,l)dl \right)^{\frac{1}{\rho}} - \nu \left(\int W_t(l) N_t(i,l)dl - \Theta_{N,t} \right)$$

where the integral is evaluated on the interval $[0,n]$, $0 \leq n \leq 1$. Henceforth we drop inessential indices for the sake of readability.[39] The first order condition with respect to $N(i,j)$ is given by[40]

$$n^{\frac{\rho-1}{\rho}} \left(\int N^{\rho}(i,l)dl \right)^{\frac{1-\rho}{\rho}} N^{\rho-1}(i,j) - \nu W(j) = 0$$

which we can rewrite as

$$\nu W(j) = n^{-\frac{(1-\rho)^2}{\rho}} N^{1-\rho}(i) \cdot N^{\rho-1}(i,j)$$

using the assumption

$$N(i) = n^{\frac{\rho-1}{\rho}} \left(\int N^{\rho}(i,l)dl \right)^{\frac{1}{\rho}}$$

Since $N(i)$ is monotonic w.r.t. $N(i,j)$ for $N(i,j) > 0$, it is clear that the constraint is binding and $\nu \neq 0$. Thus for any two households j and k, we have

$$\left(\frac{W(k)}{W(j)} \right)^{\frac{1}{\rho-1}} = \frac{N(i,k)}{N(i,j)}$$

Recall that we define the elasticity of substitution as $\epsilon = \epsilon_{W,t} = \frac{1}{1-\rho}$. By multiplying with appropriate terms and integrating, we get

$$\Theta_N = \int_0^n N(i,l)W(l)dl = \int_0^n W^{1-\epsilon}(l)dl \cdot \frac{N(i,j)}{W(j)^{-\epsilon}} \tag{152}$$

Using the definition of the wage index, given by equation (6), then equation (152) becomes

$$N(i,j) = \frac{1}{n} \frac{\Theta_N}{W} \left(\frac{W(j)}{W_H} \right)^{-\epsilon} \tag{153}$$

Inserting the last expression into the definition of aggregate labour service supplied to firm i, $N_t(i)$, we get

$$N(i) = n^{\frac{1}{1-\epsilon}} \left(\int_0^n \left[\frac{1}{n} \frac{\Theta_N}{W} \left(\frac{W(l)}{W} \right)^{-\epsilon} \right]^{\frac{\epsilon-1}{\epsilon}} dl \right)^{\frac{\epsilon}{\epsilon-1}}$$

which we can rewrite as

$$N(i) = n^{\frac{1}{1-\epsilon}-1} \frac{\Theta_N}{W^{1-\epsilon}} \cdot \left(\int_0^n W^{1-\epsilon_H}(l)dl \right)^{\frac{\epsilon}{\epsilon-1}}$$

[39] We also drop the time index since all decision on labour supply are contemporaneous.

[40] Here we employ our results from Appendix A, section 7.1.2.

which, in turn implies

$$N(i) = n^{\frac{1}{1-\epsilon}-1}\frac{\Theta_N}{W^{1-\epsilon}} \cdot \left(n^{\frac{\epsilon}{\epsilon-1}} W^{-\epsilon}\right) = \frac{\Theta_N}{W}$$

or equivalently

$$\int W(k)N(i,l)dl = \Theta_N = N(i)W \tag{154}$$

Bringing back the notation for the i-th firm employing household j at time t, and inserting the last expression into equation (153), we get

$$N_t(i,j) = \frac{1}{n}\left(\frac{W_t}{W_t(j)}\right)^{-\epsilon_{W,t}} N_t(i) \tag{155}$$

Aggregating over firms gives

$$N_t(j) = \left(\frac{W_t}{W_t(j)}\right)^{-\epsilon_{W,t}} N_t \tag{156}$$

7.3.2 Marginal cost

Let us now turn to deriving the stated expression for a generic domestic firm's marginal cost. First, to get the conditional factor demand we minimise

$$\mathcal{L} = \kappa_{g,t}(i) - \lambda(Y_{g,t}(i) - \Theta_{S,t})$$

where

$$\kappa_{g,t}(i) = \int W_t(l)N_{g,t}(i,l)dl + R_t^K K_{g,t}(i) = W_t N_{g,t}(i) + R_t^K K_{g,t}(i)$$

is the the firm's cost function and

$$Y_{g,t}(i) = K_{g,t}^{\psi_H}(i)(Z_t Z_{H,t} N_{S,t}(i))^{1-\psi_H}$$

is the firm's i production function. Again we simplify notation by dropping indices. The first order conditions w.r.t. $N(i)$ and $K(i)$ are given by[41]

$$W = \lambda(1-\psi_H)\left(\frac{K(i)}{N(i)}\right)^{\psi_H}(ZZ_H)^{1-\psi_H}$$

and

$$R^K = \lambda\psi_H\left(\frac{K(i)}{N(i)}\right)^{\psi_H-1}(ZZ_H)^{1-\psi_H}$$

which together imply

$$\frac{K(i)}{N(i)} = \frac{\psi_H}{1-\psi_H}\frac{W}{R^K} \tag{157}$$

or

$$K(i) = \frac{\psi_H}{1-\psi_H}\frac{W}{R^K}N(i)$$

[41] It is inconsequential that we do not differentiate between $N_{S,t}(i)$ and $N_t(i)$ in the derivation since $\frac{\partial N_{S,t}(i)}{\partial N_t(i)} = 1$.

Now we can derive our desired marginal cost function. Inserting the last equation into the cost function $\kappa(i)$, we get

$$\kappa(i) = N(i)W \frac{1}{1-\psi_H}$$

Marginal cost is defined as

$$MC(i) = \frac{d\kappa(i)}{dY(i)} = \frac{W}{1-\psi_H} \frac{dN(i)}{dY(i)} \tag{158}$$

and it thus suffices to find the amount of labour needed to produce an extra unit. Inserting the conditional factor demand constraint in equation (157) into the production function, we get

$$Y(i) = N(i) \left(\frac{\psi_H}{1-\psi_H}\right)^{\psi_H} \left(\frac{W}{R^K}\right)^{\psi_H} (ZZ_H)^{1-\psi_H}$$

Rearranging and differentiating $N(i)$ with respect to $Y(i)$ gives:

$$\frac{dN(i)}{dY(i)} = \left(\frac{\psi_H}{1-\psi_H}\right)^{-\psi_H} \left(\frac{W}{R^K}\right)^{-\psi_H} \frac{1}{(ZZ_H)^{1-\psi_H}}$$

Bringing back the full notation this implies

$$MC_t(i) = \frac{1}{1-\psi_H} W_t \left(\frac{\psi_H}{1-\psi_H}\right)^{-\psi_H} \left(\frac{W_t}{R_t^K}\right)^{-\psi_H} \frac{1}{(Z_t Z_{H,t})^{1-\psi_H}} \tag{159}$$

As established in previous sections all firms pay the same wage and the same rental rate of capital, thus it is clear that all firm have the same marginal cost, i.e., for all i we have

$$MC_t = MC_t(i) = \frac{1}{1-\psi_H} \left(\frac{\psi_H}{1-\psi_H}\right)^{-\psi_H} \frac{W_t^{1-\psi_H} (R_t^K)^{\psi_H}}{(Z_t Z_{H,t})^{1-\psi_H}} \tag{160}$$

7.4 Demand schedules

7.4.1 Domestic demand schedules

In deriving a domestic household's demand schedule as a function of total demand, we proceed in two steps. First, we maximise domestic and foreign consumption, separately, given expenditures on each. Subsequently, we minimise expenditures on total consumption, i.e. the sum of domestic and foreign consumption, given the total consumption level. It is clear that these are necessary conditions for a rational agent since utility is an increasing function of consumption. For the first step, we define the Lagrangian

$$\mathcal{L} = C_{l,t}(j) - \nu \left(\int P_{l,t}(k) C_{l,t}(k,j) dk - \Theta_{C_l} \right) \tag{161}$$

$$= n_l^{\frac{\rho-1}{\rho}} \left(\int C_{l,t}^{\rho}(k,j) dk \right)^{\frac{1}{\rho}} - \nu \left(\int P_{l,t}(k) C_{l,t}(k,j) dk - \Theta_{C_l} \right) \tag{162}$$

where $l \in \{H, F\}$, $n_l = n$ when $l = H$, and $n_l = 1-n$ when $l = F$, and the integral is evaluated over the corresponding interval. Now we will drop all dispensable indices for legibility, which implicitly shows that the derivation for the foreign and domestic consumption is identical. The first order condition with respect to $C(i)$ is given by[42]

$$n_l^{\frac{\rho-1}{\rho}}\left(\int C^{\rho}(i)di\right)^{\frac{1-\rho}{\rho}} C^{\rho-1}(i)-\nu P(i)=0$$

The constraint is clearly binding, implying $\nu \neq 0$. Similarly as for labour demand we have for any two goods i and k:

$$\left(\frac{P(k)}{P(i)}\right)^{\frac{1}{\rho-1}}=\frac{C(k)}{C(i)}$$

The elasticity of substitution is given by $\epsilon=\frac{1}{1-\rho}$. Cross-multiplying by the appropriate terms and integrating, we get

$$\Theta_{C_l}=\int_a^b C(k)P(k)dk=\int_a^b P^{1-\epsilon}(k)dk\cdot\frac{C(i)}{P(i)^{-\epsilon}} \tag{163}$$

where a, b are determined by whether we are considering domestic or foreign markets. In the domestic market case we have $a=0, b=n$, while we have $a=n, b=1$, in the foreign case. Bringing back the notation for the j-th household consuming good i, and using the definition of the consumption price index, given by equations (10) and (11), then equation (163) becomes

$$C_{l,t}(i,j)=\frac{1}{n_l}\frac{\Theta_{C_l}}{P_{l,t}}\left(\frac{P_{l,t}(i)}{P_{l,t}}\right)^{-\epsilon} \tag{164}$$

Inserting the last expression into the definition of aggregate consumption of household j, $C_{l,t}(j)$, we get

$$C_{l,t}(j)=n_l^{\frac{1}{1-\epsilon}}\left(\int_a^b\left[\frac{1}{n_l}\frac{\Theta_{C_l}}{P_{l,t}}\left(\frac{P_{l,t}(k)}{P_{l,t}}\right)^{-\epsilon}\right]^{\frac{\epsilon-1}{\epsilon}}dk\right)^{\frac{\epsilon}{\epsilon-1}}$$

which we can rewrite as

$$C_{l,t}(j)=n_l^{\frac{1}{1-\epsilon}-1}\frac{\Theta_{C_l}}{P_{l,t}^{1-\epsilon}}\cdot\left(\int_a^b P_{l,t}^{1-\epsilon_{l,t}}(k)dk\right)^{\frac{\epsilon}{\epsilon-1}}$$

which, in turn implies

$$C_{l,t}(j)=n_l^{\frac{1}{1-\epsilon}-1}\frac{\Theta_{C_l}}{P_{l,t}^{1-\epsilon}}\cdot\left(n_l^{\frac{\epsilon}{\epsilon-1}}P_{l,t}^{-\epsilon}\right)=\frac{\Theta_{C_l}}{P_{l,t}}$$

or equivalently

$$\Theta_{C_l}=C_{l,t}(j)P_{l,t}$$

By inserting the last expression into equation (164) we get

$$C_{l,t}(i,j)=\frac{1}{n_l}\frac{C_{l,t}(j)P_{l,t}}{P_{l,t}}\left(\frac{P_{l,t}(i)}{P_{l,t}}\right)^{-\epsilon_{l,t}}$$

which implies

$$C_{l,t}(i,j)=\frac{1}{n_l}\left(\frac{P_{l,t}(i)}{P_{l,t}}\right)^{-\epsilon_{l,t}}C_{l,t}(j) \tag{165}$$

[42]Here we use our results on functional derivatives from Appendix A, section 7.1.2.

representing domestic household j's demand schedule for a domestic good i in terms of relative prices and household j's aggregate demand schedule.

Next, we minimise total expenditure for a given level of total consumption, with respect to consumption of domestic goods and imports. We have the corresponding Lagrangian:

$$\begin{aligned}\mathcal{L} = &\int P_{H,t}(k)C_{H,t}(k,j)dk + \int P_{F,t}(k)C_{F,t}(k,j)dk \\ &- \lambda\left(\left[\bar{\alpha}^{\frac{1}{\eta}}C_{H,t}^{\frac{\eta-1}{\eta}}(j) + (1-\bar{\alpha})^{\frac{1}{\eta}}C_{F,t}^{\frac{\eta-1}{\eta}}(j)\right]^{\frac{\eta}{\eta-1}} - \Theta_C\right)\end{aligned}$$

Above we demonstrated that

$$\int P_{l,t}(k)C_{l,t}(k,j)dk = \Theta_{C_l} = C_{l,t}(j)P_{l,t}$$

and we can thus write the Lagrangian as

$$\mathcal{L} = P_{H,t}C_{H,t}(j) + P_{F,t}C_{F,t}(j) - \lambda\left(\left[\bar{\alpha}^{\frac{1}{\eta}}C_{H,t}^{\frac{\eta-1}{\eta}}(j) + (1-\bar{\alpha})^{\frac{1}{\eta}}C_{F,t}^{\frac{\eta-1}{\eta}}(j)\right]^{\frac{\eta}{\eta-1}} - \Theta_C\right)$$

First order conditions are given by

$$P_{H,t} = \lambda\bar{\alpha}^{\frac{1}{\eta}}\left(\frac{C_t(j)}{C_{H,t}(j)}\right)^{\frac{1}{\eta}}$$

$$P_{F,t} = \lambda(1-\bar{\alpha})^{\frac{1}{\eta}}\left(\frac{C_t(j)}{C_{F,t}(j)}\right)^{\frac{1}{\eta}}$$

which we will find useful to rewrite as

$$C_{H,t}(j) = \lambda^{\eta}\bar{\alpha}P_{H,t}^{-\eta}C_t(j) \tag{166}$$

$$C_{F,t}(j) = \lambda^{\eta}(1-\bar{\alpha})P_{F,t}^{-\eta}C_t(j) \tag{167}$$

Together, equations (166) and (167) imply

$$\frac{C_{H,t}(j)}{C_{F,t}(j)} = \frac{\bar{\alpha}}{1-\bar{\alpha}}\left(\frac{P_{H,t}}{P_{F,t}}\right)^{-\eta}$$

$$\lambda = \bar{\alpha}^{-\frac{1}{\eta}}P_H\left(\frac{C_t(j)}{C_{H,t}(j)}\right)^{-\frac{1}{\eta}}$$

We will use the last two equations to show that $\lambda = P_t$, a result we anticipate given the interpretation of the Lagrangian multiplier as the shadow cost of increasing total consumption. Multiplying the latter by $C_t(j)$, and using the former to express $C_{H,t}(j)$ in terms of $C_{F,t}(j)$, we get

$$
\begin{aligned}
\lambda \cdot C_t &= P_{H,t}\bar{\alpha}^{\frac{-1}{\eta}} C_{H,t}^{\frac{1}{\eta}} C_t^{\frac{\eta-1}{\eta}} \\
&= P_{H,t}\bar{\alpha}^{\frac{-1}{\eta}} C_{H,t}^{\frac{1}{\eta}} \left[\bar{\alpha}^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + (1-\bar{\alpha})^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}}\right] \\
&= P_{H,t}C_H + P_H\bar{\alpha}^{\frac{-1}{\eta}} \left(C_{F,t}\frac{\bar{\alpha}}{1-\bar{\alpha}}\left(\frac{P_{H,t}}{P_{F,t}}\right)^{-\eta}\right)^{\frac{1}{\eta}} (1-\bar{\alpha})^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}} \\
&= P_{H,t}C_{H,t} + P_{F,t}C_{F,t}
\end{aligned}
\tag{168}
$$

where we have suppressed the reference to the j-th household. Bringing back the full notation, we can write

$$
\lambda \cdot C_t(j) = P_{H,t}C_{H,t}(j) + P_{F,t}C_{F,t}(j)
$$

Multiplying both sides by $\frac{1}{n}$ and integrating over the domestic economy we see that $P_t = \lambda$. Now, using the envelope theorem, we have that

$$
\lambda = P_{H,t}\frac{\delta C_{H,t}(j)}{\delta \Theta_C} + P_{F,t}\frac{\delta C_{F,t}(j)}{\delta \Theta_C}
$$

At the optimum we have $\Theta_C = C_t(j)$, hence we have by equations (166) and (167):

$$
\frac{\delta C_{H,t}(j)}{\delta \Theta_C} = \lambda^{\eta}\bar{\alpha}P_{H,t}^{-\eta}
$$

and

$$
\frac{\delta C_{F,t}(j)}{\delta \Theta_C} = \lambda^{\eta}(1-\bar{\alpha})P_{F,t}^{-\eta}
$$

Inserting these into the expression just derived for λ, we get

$$
P_t = \lambda = \left[\bar{\alpha}^{\frac{1}{\eta}} P_{H,t}^{1-\eta} + (1-\bar{\alpha})P_{F,t}^{1-\eta}\right]^{\frac{1}{1-\eta}}
$$

as foretold by the main text. Equations (166) and (167) become

$$
C_{H,t}(j) = \bar{\alpha}\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t(j)
$$

$$
C_{F,t}(j) = (1-\bar{\alpha})\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t(j)
$$

Inserting the former into equation (165), we get

$$
C_{H,t}(i,j) = \frac{1}{n}\left(\frac{P_{H,t}}{P_{H,t}(i)}\right)^{-\epsilon_{H,t}} \bar{\alpha}\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t(j)
$$

Aggregating over domestic households $j \in [0, n]$ gives us the domestic consumption demand for good i as

$$
\begin{aligned}
C_{H,t}(i) &= \int_0^n C_{H,t}(i,j)dj \\
&= \bar{\alpha} \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\epsilon_{H,t}} \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} \left(\frac{1}{n} \right) \int_0^n C_t(j)dj \\
&= \bar{\alpha} \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\epsilon_{H,t}} \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} C_t
\end{aligned}
$$

Using $C_t = \left(\frac{1}{n}\right) \int_0^n C_t(j)dj$, the total consumption per capita.

Similarly, we get the domestic demand for a foreign producer i's good:

$$
C_{F,t}(i) = (1 - \bar{\alpha}) \left(\frac{P_{F,t}(i)}{P_{F,t}} \right)^{-\epsilon_{F,t}} \left(\frac{P_{F,t}}{P_t} \right)^{-\eta} \frac{n}{1-n} C_t
$$

7.4.2 Foreign demand schedules

We need not trouble ourselves with the foreign consumption of foreign goods but we will derive the demand for domestic exports. The case of exports differs from the domestic demand schedules since we assume the existence of an export adjustment cost and specialised export firms. We define the export adjustment cost as

$$
\Gamma^*_{H,t} = \frac{\theta^*_M}{2} \left(\frac{C^*_{H,t}}{C^*_t} \frac{C^*_{t-1}}{C^*_{H,t-1}} - 1 \right)^2
$$

which enters the foreign consumption basket through

$$
C^*_t(j) = \left[(\bar{\alpha}^*)^{\frac{1}{\eta}} \check{C}^*_{H,t}(j)^{\frac{\eta-1}{\eta}} + (1-\bar{\alpha}^*)^{\frac{1}{\eta}} C^*_{F,t}(j)^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \tag{169}
$$

where

$$
\check{C}^*_{H,t}(j) = [1 - \Gamma^*_{H,t}] C^*_{H,t}(j) \tag{170}
$$

and

$$
C^*_{H,t}(j) = \left[(1-\alpha_E)^{\frac{1}{\eta_E}} C^*_{g,t}(j)^{\frac{\eta_E - 1}{\eta_E}} + \alpha_E^{\frac{1}{\eta_E}} C^*_{E,t}(j)^{\frac{\eta_E - 1}{\eta_E}} \right]^{\frac{\eta_E}{\eta_E - 1}} \tag{171}
$$

Export adjustment cost is included to reflect the marketing cost of entering a new market, the cost of expanding a geographical network, etc. Employing the methods in the preceding section we can easily get the expression:

$$
C^*_{g,t}(i,j) = \left(\frac{1}{n} \right) \left(\frac{P^*_{g,t}(i)}{P^*_{g,t}} \right)^{-\epsilon_{H,t}} C^*_{g,t}(j) \tag{172}
$$

Since we assume that $P^*_{E,t}(i) = P^*_{g,t}$, we clearly have

$$C^*_{E,t}(i,j) = \left(\frac{1}{n}\right) C^*_{E,t}(j) \tag{173}$$

Before we derive demand of domestic and foreign goods as a function of total demand we seek to express the choice between the generic export goods and specialised export goods. To that end we minimise expenditure on each good, given foreign consumption level for domestic goods.

We thus minimise the Lagrangian[43]

$$\begin{aligned}\mathcal{L} = {} & P^*_{H,t}(C^*_{g,t}(j) + C^*_{E,t}(j)) \\ & - \lambda^*_{H,t}\left(\left[\alpha_E^{\frac{1}{\eta_E}}(C^*_{E,t}(j))^{\frac{\eta_E-1}{\eta_E}} + (1-\alpha_E)^{\frac{1}{\eta_E}}(C^*_{g,t}(j))^{\frac{\eta_E-1}{\eta_E}}\right]^{\frac{\eta_E}{\eta_E-1}} - \phi^*_H\right)\end{aligned}$$

where we use $P^*_{H,t} = P^*_{g,t} = P^*_{E,t}$. Omitting inessential notation, the first order conditions can be written as

$$\begin{aligned}\lambda^*_{H,t} &= P^*_{H,t}\alpha_E^{-\frac{1}{\eta_E}}\left(\frac{C^*_{H,t}}{C^*_{E,t}}\right)^{-\frac{1}{\eta_E}} \\ \lambda^*_{H,t} &= P^*_{H,t}(1-\alpha_E)^{-\frac{1}{\eta_E}}\left(\frac{C^*_{H,t}}{C^*_{g,t}}\right)^{-\frac{1}{\eta_E}} \\ C^*_{H,t} &= \phi^*_H\end{aligned}$$

Note that together these equations imply

$$\frac{\alpha_E}{1-\alpha_E} = \frac{C^*_{E,t}}{C^*_{g,t}} \tag{174}$$

Furthermore, we have

$$\begin{aligned}\lambda^*_{H,t}C^*_{H,t} &= P^*_{H,t}\alpha_E^{-\frac{1}{\eta_E}}\left(\frac{C^*_{E,t}}{C^*_{H,t}}\right)^{\frac{1}{\eta_E}} C^*_{H,t} \\ &= P^*_{H,t}\alpha_E^{-\frac{1}{\eta_E}}(C^*_{E,t})^{\frac{1}{\eta_E}}(C^*_{H,t})^{\frac{\eta_E-1}{\eta_E}} \\ &= P^*_{H,t}\alpha_E^{-\frac{1}{\eta_E}}(C^*_{E,t})^{\frac{1}{\eta_E}}\left[\alpha_E^{\frac{1}{\eta_E}}(C^*_{E,t}(j))^{\frac{\eta_E-1}{\eta_E}} + (1-\alpha_E)^{\frac{1}{\eta_E}}(C^*_{g,t})^{\frac{\eta_E-1}{\eta_E}}\right] \\ &= P^*_{H,t}C^*_{E,t} + P^*_{H,t}\alpha_E^{-\frac{1}{\eta_E}}(C^*_{g,t})^{\frac{1}{\eta_E}}\left(\frac{\alpha_E}{1-\alpha_E}\right)^{\frac{1}{\alpha_E}}(1-\alpha_E)^{\frac{1}{\eta_E}}(C^*_{g,t})^{\frac{\eta_E-1}{\eta_E}} \\ &= P^*_{H,t}C^*_{E,t} + P^*_{H,t}C^*_{g,t}\end{aligned}$$

Using the envelope theorem and similar arguments as in the preceding section we deduce that $\lambda^*_{H,t} = P^*_{H,t}$ and we get, unsurprisingly:

[43] Again relying on the methods of functional derivatives outlined in Appendix A, section 7.1.2.

$$C^*_{E,t}(j) = \alpha_E C^*_{H,t}(j) \tag{175}$$
$$C^*_{g,t}(j) = (1-\alpha_E)C^*_{H,t}(j) \tag{176}$$
$$C^*_{H,t}(j) = C_{E,t}(j) + C_{g,t}(j) \tag{177}$$

Combining this result with equation (172) and (173) We can write

$$C^*_{g,t}(i,j) = (1-\alpha_E)\left(\frac{1}{1-n}\right)\left(\frac{P^*_{H,t}(i)}{P^*_{H,t}}\right)^{-\epsilon_{H,t}} C^*_{H,t}(j) \tag{178}$$

$$C^*_{E,t}(i,j) = \alpha_E\left(\frac{1}{n}\right)C^*_{H,t}(j) \tag{179}$$

Finally we want to express the demand schedules in terms of total demand. We thus minimise the Lagrangian

$$\begin{aligned}\mathcal{L} = {} & P^*_{H,t}C^*_{H,t}(j) + P^*_{F,t}C^*_{F,t}(j) \\ & - \lambda^*_t\left(\left[(\bar{\alpha}^*)^{\frac{1}{\eta^*}}(\check{C}^*_{H,t}(j))^{\frac{\eta^*-1}{\eta^*}} + (1-\bar{\alpha^*})^{\frac{1}{\eta^*}}(C^*_{F,t})^{\frac{\eta^*-1}{\eta^*}}\right]^{\frac{\eta^*}{\eta^*-1}} - \phi^*\right)\end{aligned}$$

And the first order conditions are

$$0 = P^*_{H,t} - \lambda^*_t\frac{\partial C^*_t(j)}{\partial C^*_{H,t}(j)} \tag{180}$$
$$0 = P^*_{F,t} - \lambda^*_t\frac{\partial C^*_t(j)}{\partial C^*_{F,t}(j)} \tag{181}$$

By the chain rule we have

$$\frac{\partial C^*_t(j)}{\partial C^*_{H,t}(j)} = \frac{\partial C^*_t(j)}{\partial \check{C}^*_{H,t}(j)}\frac{\partial \check{C}^*_{H,t}(j)}{\partial C^*_{H,t}(j)}$$

and by definition

$$\frac{\partial \check{C}^*_{H,t}(j)}{\partial C^*_{H,t}(j)} = [1-\Gamma^*_{H,t}] - \frac{\partial \Gamma^*_{H,t}}{\partial C^*_{H,t}(j)}C^*_{H,t}(j)$$

Thus

$$\begin{aligned}\frac{\partial C^*_t(j)}{\partial C^*_{H,t}(j)} &= (\bar{\alpha^*})^{\frac{1}{\eta^*}}\left(\frac{C^*_t(j)}{\check{C}^*_{H,t}(j)}\right)^{\frac{1}{\eta^*}}\frac{\partial \check{C}^*_{H,t}(j)}{\partial C^*_{H,t}(j)} \\ &= (\bar{\alpha^*})^{\frac{1}{\eta^*}}\left(\frac{C^*_t(j)}{C^*_{H,t}(j)[1-\Gamma^*_{H,t}]}\right)^{\frac{1}{\eta^*}}\frac{\partial \check{C}^*_{H,t}(j)}{\partial C^*_{H,t}(j)} \\ &= (\bar{\alpha^*})^{\frac{1}{\eta^*}}\left(\frac{C^*_t(j)}{C^*_{H,t}(j)}\right)^{\frac{1}{\eta^*}}\check{\Gamma}_{H,t}\end{aligned}$$

where we have defined

$$\check{\Gamma}^*_{H,t} = \frac{1 - \Gamma^*_{H,t} - \frac{\partial \Gamma^*_{H,t}}{\partial C_{H,t}(j)} C^*_{H,t}(j)}{(1 - \Gamma^*_{H,t})^{\frac{1}{\eta^*}}}$$

With respect to foreign demand of foreign goods we have

$$\frac{\partial C^*_t(j)}{\partial C^*_{F,t}(j)} = (1 - \bar{\alpha^*})^{\frac{1}{\eta^*}} \left(\frac{C^*_t(j)}{C^*_{F,t}(j)} \right)^{\frac{1}{\eta^*}}$$

We can rewrite these conditions as

$$\lambda^*_t = (1 - \bar{\alpha})^{-\frac{1}{\eta^*}} P^*_{F,t} \left(\frac{C^*_t(j)}{C^*_{F,t}(j)} \right)^{-\frac{1}{\eta^*}} = (\bar{\alpha})^{-\frac{1}{\eta^*}} P^*_{H,t} \left(\frac{C^*_t(j)}{C^*_{H,t}(j)} \right)^{-\frac{1}{\eta^*}} (\check{\Gamma}^*_{H,t})^{-1} \tag{182}$$

and

$$(C^*_{F,t}(j))^{\frac{1}{\eta^*}} = \left(\frac{\bar{\alpha}}{1 - \bar{\alpha}} \right)^{-\frac{1}{\eta^*}} \frac{P^*_{H,t}}{P^*_{F,t}} (C^*_{H,t}(j))^{\frac{1}{\eta^*}} (\check{\Gamma}^*_{H,t})^{-1} \tag{183}$$

Using the three preceding equations we follow the same procedure as before and seek to express λ^*_t in terms of prices. We have

$$\begin{aligned}
\lambda^*_t \cdot C^*_t(j) &= (1 - \bar{\alpha})^{-\frac{1}{\eta^*}} P^*_{F,t} (C^*_{F,t}(j))^{\frac{1}{\eta^*}} (C^*_t(j))^{\frac{\eta^*-1}{\eta^*}} \\
&= (1 - \bar{\alpha})^{-\frac{1}{\eta^*}} P^*_{F,t} (C^*_{F,t}(j))^{\frac{1}{\eta^*}} \left[\bar{\alpha}^{\frac{1}{\eta^*}} \check{C}_{H,t}(j)^{\frac{\eta^*-1}{\eta^*}} + (1 - \bar{\alpha})^{\frac{1}{\eta^*}} C_{F,t}^{\frac{\eta^*-1}{\eta^*}}(j) \right] \\
&= (1 - \bar{\alpha})^{-\frac{1}{\eta^*}} P^*_{F,t} (C^*_{F,t}(j))^{\frac{1}{\eta^*}} \bar{\alpha}^{\frac{1}{\eta^*}} \check{C}_{H,t}(j)^{\frac{\eta^*-1}{\eta^*}} \\
&\quad + (1 - \bar{\alpha})^{-\frac{1}{\eta^*}} P^*_{F,t} (C^*_{F,t}(j))^{\frac{1}{\eta^*}} (1 - \bar{\alpha})^{\frac{1}{\eta^*}} C_{F,t}(j)^{\frac{\eta^*-1}{\eta^*}} \\
&= (1 - \bar{\alpha})^{-\frac{1}{\eta^*}} P^*_{F,t} \left(\frac{\bar{\alpha}}{1 - \bar{\alpha}} \right)^{-\frac{1}{\eta^*}} \frac{P^*_{H,t}}{P^*_{F,t}} (C^*_{H,t}(j))^{\frac{1}{\eta^*}} (\check{\Gamma}^*_{H,t})^{-1} \bar{\alpha}^{\frac{1}{\eta^*}} \check{C}_{H,t}(j)^{\frac{\eta^*-1}{\eta^*}} \\
&\quad + (1 - \bar{\alpha})^{-\frac{1}{\eta^*}} P^*_{F,t} (C^*_{F,t}(j))^{\frac{1}{\eta^*}} (1 - \bar{\alpha})^{\frac{1}{\eta^*}} C_{F,t}(j)^{\frac{\eta^*-1}{\eta^*}} \\
&= P^*_{H,t} C^*_{H,t}(j) (\check{\Gamma}^*_{H,t})^{-1} [1 - \Gamma^*_{H,t}]^{\frac{\eta^*-1}{\eta^*}} + P^*_{F,t} C^*_{F,t}(j)
\end{aligned}$$

Alternatively we can write

$$\begin{aligned}
(\check{\Gamma}^*_{H,t})^{-1} [1 - \Gamma^*_{H,t}]^{\frac{\eta^*-1}{\eta^*}} &= \left(\frac{1 - \Gamma^*_{H,t} - \frac{\partial \Gamma^*_{H,t}}{\partial C_{H,t}(j)} C^*_{H,t}}{(1 - \Gamma^*_{H,t})^{\frac{1}{\eta^*}}} \right)^{-1} [1 - \Gamma^*_{H,t}]^{\frac{\eta^*-1}{\eta^*}} \\
&= \frac{1 - \Gamma^*_{H,t}}{1 - \Gamma^*_{H,t} - \frac{\partial \Gamma^*_{H,t}}{\partial C_{H,t}(j)} C^*_{H,t}} \\
&= \hat{\Gamma}^*_{H,t}
\end{aligned}$$

which gives

$$\lambda^*_t \cdot C^*_t(j) = P^*_{H,t} C^*_{H,t}(j) \hat{\Gamma}^*_{H,t} + P^*_{F,t} C^*_{F,t}(j) \tag{184}$$

$$= P^*_{H,t} \left(\frac{\partial \check{C}^*_{H,t}}{\partial C^*_{H,t}} \right)^{-1} \check{C}^*_{H,t} + P^*_{F,t} C^*_{F,t} \tag{185}$$

$$= \check{P}^*_{H,t} \check{C}^*_{H,t} + P^*_{F,t} C^*_{F,t} \tag{186}$$

If we define

$$\check{P}^*_{H,t} = P^*_{H,t}\left(\frac{\partial \check{C}^*_{H,t}}{\partial C^*_{H,t}}\right)^{-1}$$

Transforming both sides to economy wide per capita terms, we get $\lambda^*_t = P^*_t$ by definition. We can also represent P^*_t in terms of $P^*_{F,t}$ and $P^*_{H,t}$. To that end we use the envelope theorem yet again and get

$$\lambda^*_t = P^*_{H,t}\frac{\partial C^*_{H,t}(j)\hat{\Gamma}^*_{H,t}}{\partial C^*_t(j)} + P^*_{F,t}\frac{\partial C^*_{F,t}(j)}{\partial C^*_t(j)} \tag{187}$$

From the first order conditions we see that

$$\frac{\partial C^*_{F,t}(j)}{\partial C^*_t} = (\lambda^*_t)^{\eta^*}(1-\bar{\alpha})(P^*_{F,t})^{-\eta^*}$$

For exports we have

$$C^*_{H,t}(j)\hat{\Gamma}^*_{H,t} = \left(\frac{\lambda^*_t}{P^*_{H,t}}\right)^{\eta^*}\overline{\alpha^*}C^*_t(j)\left(\frac{\partial \check{C}^*_{H,t}(j)}{\partial C^*_{H,t}(j)}\right)^{\eta^*-1}$$

and thus

$$\frac{\partial (C^*_{H,t}(j)\hat{\Gamma}^*_{H,t})}{\partial C^*_t(j)} = \left(\frac{\lambda^*_t}{P^*_{H,t}}\right)^{\eta^*}\overline{\alpha^*}\frac{\partial}{\partial C^*_t(j)}\left(C^*_t(j)\left(\frac{\partial \check{C}^*_{H,t}(j)}{\partial C^*_{H,t}(j)}\right)^{\eta^*-1}\right)$$

from which we get

$$\begin{aligned}\lambda^*_t = P^*_{H,t}\left(\frac{\lambda^*_t}{P^*_{H,t}}\right)^{\eta^*}\overline{\alpha^*}\frac{\partial}{\partial C^*_t(j)}\left(C^*_t(j)\left(\frac{\partial \check{C}^*_{H,t}(j)}{\partial C^*_{H,t}(j)}\right)^{\eta^*-1}\right)\\ + P^*_{F,t}\left(\frac{\lambda^*_t}{P^*_{F,t}}\right)^{\eta^*}(1-\overline{\alpha^*})\end{aligned}$$

Denote

$$g_t(\cdot) = \frac{\partial}{\partial C^*_t(j)}\left(C^*_t(j)\left(\frac{\partial \check{C}^*_{H,t}(j)}{\partial C^*_{H,t}(j)}\right)^{\eta^*-1}\right)$$

and we get

$$P^*_t = \lambda^*_t = \left[\bar{\alpha}^*(P^*_{H,t})^{1-\eta}g_t(\cdot) + (1-\bar{\alpha}^*)(P^*_{F,t})^{1-\eta}\right]^{\frac{1}{1-\eta}} \tag{188}$$

Using $P^*_t = \lambda^*_t$ with the first order conditions, we get

$$C^*_{H,t}(j) = \bar{\alpha}^*\left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta}(\check{\Gamma}^*_{H,t}(j))^{\eta}\cdot C^*_t(j) \tag{189}$$

and thus, using equation (178) and (179), we have

$$C^*_{g,t}(i,j) = (1-\alpha_E)\bar{\alpha}^*\left(\frac{P^*_{H,t}(i)}{P^*_{H,t}}\right)^{-\epsilon_{H,t}}\left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta}(\check{\Gamma}^*_{H,t}(j))^{\eta}\cdot\left(\frac{1}{n}\right)C^*_t(j) \tag{190}$$

and

$$C^*_{E,t}(i,j) = \alpha_E \bar{\alpha}^* \left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta} (\check{\Gamma}^*_{H,t}(j))^\eta \cdot \left(\frac{1}{n}\right) C^*_t(j) \tag{191}$$

And further, integrating with respect to j, over the appropriate interval, we get

$$C^*_{g,t}(i) = (1-\alpha_E)\left(\frac{1-n}{n}\right)\bar{\alpha}^* \left(\frac{P^*_{H,t}(i)}{P^*_{H,t}}\right)^{-\epsilon_{H,t}} \left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta} \hat{C}^*_t \tag{192}$$

$$C^*_{E,t}(i) = \alpha_E \left(\frac{1-n}{n}\right)\bar{\alpha}^* \left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta} \hat{C}^*_t \tag{193}$$

where $\hat{C}^*_t = \frac{1}{1-n}\int_n^1 (\check{\Gamma}^*_{H,t}(j))^\eta \cdot C^*_t(j)\, dj$.

7.5 Market equilibrium

Since the final good can be transformed one-to-one into any type of good, i.e. investment, consumption, or public consumption good, it follows that the elasticity of substitution between goods of any type are the same regardless of the use of the final good. Moreover, the export adjustment cost is independent of the final use of the final good since it only depends on relative quantities, which in turn depend on the elasticities of substitution and relative prices. Consequently, we may derive demand relations for the good produced by any firm i, corresponding to the private consumption demand relations above for public consumption, investment and maintenance of machinery. Lastly, the equilibrium of the model is symmetric, i.e. every generic firm chooses the same price given the chance and each domestic agent chooses the same consumption basket. This follows from our earlier discussion on the implication of the complete markets assumption and the fact that every generic firm has the same marginal cost.

7.5.1 Market clearing of domestic goods

Demand for the two types of goods sold home and abroad are described by the three equations:

$$\begin{aligned}
X_{H,t}(i) &= \bar{\alpha}\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon_{H,t}}\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} X_t \\
X^*_{g,t}(i) &= \bar{\alpha}^*(1-\alpha_E)\left(\frac{1-n}{n}\right)\left(\frac{P^*_{H,t}(i)}{P^*_{H,t}}\right)^{-\varepsilon^*_{H,t}}\left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta} \\
&\quad \times \left(\frac{1}{1-n}\right)\int_n^1 \frac{\left(1-\Gamma^*_{H,t} - X^*_{H,t}(j)\frac{\partial \Gamma^*_{H,t}}{\partial X^*_{H,t}(j)}\right)^\eta}{1-\Gamma^*_{H,t}} X^*_t(j)dj
\end{aligned}$$

$$
\begin{aligned}
X^*_{E,t}(i) = {} & \bar{\alpha}^*\alpha_E\left(\frac{1-n}{n}\right)\left(\frac{P^*_{H,t}(i)}{P^*_{H,t}}\right)^{-\varepsilon^*_{H,t}} \\
& \times\left(\frac{1}{1-n}\right)\int_n^1 \frac{\left(1-\Gamma^*_{H,t}-X^*_{H,t}(j)\frac{\partial\Gamma^*_{H,t}}{\partial X^*_{H,t}(j)}\right)^\eta}{1-\Gamma^*_{H,t}}X^*_t(j)dj
\end{aligned}
$$

where $X \in \{C, I, G, M\}$. In equilibrium all agents choose the same action and we can suppose that

$$
\left(\frac{1}{1-n}\right)\int_n^1 \frac{\left(1-\Gamma^*_{H,t}-X^*_{H,t}(j)\frac{\partial\Gamma^*_{H,t}}{\partial X^*_{H,t}}\right)^\eta}{1-\Gamma^*_{H,t}(j)}X^*_t(j)dj = \frac{\left(1-\Gamma^*_{H,t}-X^*_{H,t}\frac{\partial\Gamma^*_{H,t}}{\partial X^*_{H,t}}\right)^\eta}{1-\Gamma^*_{H,t}}X^*_t
$$

and we get

$$
\begin{aligned}
A_{H,t}(i) &= \bar{\alpha}\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon_{H,t}}\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}A_t \\
A^*_{g,t}(i) &= \bar{\alpha}^*(1-\alpha_E)\left(\frac{1-n}{n}\right)\left(\frac{P^*_{H,t}(i)}{P^*_{H,t}}\right)^{-\varepsilon^*_{H,t}}\left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta}\frac{\left(1-\Gamma^*_{H,t}-A^*_{H,t}\frac{\partial\Gamma^*_{H,t}}{\partial A^*_{H,t}}\right)^\eta}{1-\Gamma^*_{H,t}}A^*_t \\
A^*_{E,t}(i) &= \bar{\alpha}^*\alpha_E\left(\frac{1-n}{n}\right)\left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta}\frac{\left(1-\Gamma^*_{H,t}-A^*_{H,t}\frac{\partial\Gamma^*_{H,t}}{\partial A^*_{H,t}}\right)^\eta}{1-\Gamma^*_{H,t}}A^*_t
\end{aligned}
$$

where

$$
\begin{aligned}
A_{H,t}(i) &= C_{H,t}(i) + I_{H,t}(i) + M_{H,t}(i) + G_{H,t}(i) \\
A^*_{H,t}(i) &= A^*_{g,t}(i) + A^*_{E,t} \\
A^*_{l,t}(i) &= C^*_{l,t}(i) + I^*_{l,t}(i) + M^*_{l,t}(i) + G^*_{l,t}(i)
\end{aligned}
$$

with $l \in \{g, E\}$. Recall that we have defined

$$
A_t = C_t + I_t + G_t + M_t \quad \text{and} \quad A^*_t = C^*_t + I^*_t + G^*_t + M^*_t
$$

Note that $A^*_{E,t}(i)$ is independent of i, and we can safely write $A^*_{E,t}$. Aggregate supply is defined as

$$
\begin{aligned}
Y^h_{H,t} &= \left(\frac{1}{n}\int_0^n Y_{H,t}^{\frac{\varepsilon_{H,t}-1}{\varepsilon_{H,t}}}(i)di\right)^{\frac{\varepsilon_{H,t}}{\varepsilon_{H,t}-1}} \\
Y^f_{g,t} &= \left(\frac{1}{n}\int_0^n (Y^*_{g,t}(i))^{\frac{\varepsilon^*_{H,t}-1}{\varepsilon^*_{H,t}}}di\right)^{\frac{\varepsilon^*_{H,t}}{\varepsilon^*_{H,t}-1}} \\
Y^f_{E,t} &= \frac{1}{n}\int_0^n Y^*_{E,t}(i)di
\end{aligned}
$$

Before we equate supply and demand for good i by setting $A_{H,t}(i) = Y_{H,t}(i)$, $A_{g,t}(i) = Y^*_{g,t}(i)$, $A_{E,t}(i) = Y^*_{E,t}(i)$, and derive aggregate demand in equilibrium, recall that by definition we have

$$P_{H,t}^{1-\varepsilon_{H,t}} = \frac{1}{n}\int_0^n P_{H,t}^{1-\varepsilon_{H,t}}(i)di \quad \text{and} \quad (P^*_{H,t})^{1-\varepsilon^*_{H,t}} = \frac{1}{n}\int_0^n (P^*_{H,t}(i))^{1-\varepsilon^*_{H,t}}(i)di$$

and we can thus write

$$\left(\frac{1}{n}\int_0^n (P_{H,t}^{-\varepsilon_{H,t}}(i))^{\frac{\varepsilon_{H,t}-1}{\varepsilon_{H,t}}} di\right)^{\frac{\varepsilon_{H,t}}{\varepsilon_{H,t}-1}} = P_{H,t}^{-\varepsilon_{H,t}} \tag{194}$$

$$\left(\frac{1}{n}\int_0^n ((P^*_{H,t}(i))^{-\varepsilon^*_{H,t}})^{\frac{\varepsilon^*_{H,t}-1}{\varepsilon^*_{H,t}}} di\right)^{\frac{\varepsilon^*_{H,t}}{\varepsilon^*_{H,t}-1}} = (P^*_{H,t})^{-\varepsilon^*_{H,t}} \tag{195}$$

$$\tag{196}$$

Further recall that we define $1 - \bar{\alpha} = (1-n)\alpha$ and $\bar{\alpha^*} = n\alpha^*$ and hence

$$\lim_{n\to 0} \frac{1-n}{n}\bar{\alpha}^* = \alpha^*, \quad \lim_{n\to 0} \bar{\alpha} = 1-\alpha \tag{197}$$

Thus aggregate domestic demand for domestic goods can be written as

$$Y^h_{H,t} = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} A_t \tag{198}$$

Aggregate foreign demand for a generic export good can be written

$$Y^f_{g,t} = \alpha^*(1-\alpha_E)\left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta} \frac{\left(1-\Gamma^*_{H,t} - A^*_{H,t}\frac{\partial\Gamma^*_{H,t}}{\partial A^*_{H,t}}\right)^{\eta}}{1-\Gamma^*_{H,t}} A^*_t \tag{199}$$

The aggregate foreign demand for the specialised export good can be greater than feasible supply and thus market equilibrium is the minimum of the aggregate demand and the upper limit of the aggregate supply:

$$Y^f_{E,t} = \min\left(A^*_{E,t}, \max(Y_{E,t})\right) \tag{200}$$

where

$$\begin{aligned} A^*_{E,t} &= \alpha^*\alpha_E\left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta} \frac{\left(1-\Gamma^*_{H,t} - A^*_{H,t}\frac{\partial\Gamma^*_{H,t}}{\partial A^*_{H,t}}\right)^{\eta}}{1-\Gamma^*_{H,t}} A^*_t \\ &= Y^f_{g,t}\frac{\alpha_E}{1-\alpha_E} \end{aligned}$$

The aggregate demand for domestic goods in equilibrium is then defined as

$$Y_{H,t} = Y^h_{H,t} + EX_t \tag{201}$$

where

$$EX_t = Y^f_{g,t} + Y^f_{E,t}$$

7.5.2 Market clearing of foreign goods

By exactly the same methods as above we get from the demand schedule for foreign goods

$$\begin{aligned} A_{F,t}(i) &= (1-\bar{\alpha})\left(\frac{n}{1-n}\right)\left(\frac{P_{F,t}(i)}{P_{F,t}}\right)^{-\varepsilon_{F,t}}\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} A_t \\ &= n\alpha\left(\frac{P_{F,t}(i)}{P_{F,t}}\right)^{-\varepsilon_{F,t}}\left(\frac{P_{F.t}}{P_t}\right)^{-\eta} A_t \end{aligned}$$

In equilibrium we have $Y^h_{F,t}(i) = A_{F,t}(i)$ thus for imports we get

$$\begin{aligned} IM_t &= \frac{1}{n}\left(\left(\frac{1}{1-n}\right)^{\frac{1}{\varepsilon_{F,t}}}\int_n^1 A_{F,t}(i)^{\frac{\varepsilon_{F,t}-1}{\varepsilon_{F,t}}}di\right)^{\frac{\varepsilon_{F,t}}{\varepsilon_{F,t}-1}} \\ &= \frac{1}{n}\left(n\alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta}P_{F,t}^{\varepsilon_{F,t}}A_t\right)\left(\left(\frac{1}{1-n}\right)^{\frac{1}{\varepsilon_{F,t}}}\int_n^1 P_{F,t}(i)^{1-\varepsilon_{F,t}}di\right)^{\frac{\varepsilon_{F,t}}{\varepsilon_{F,t}-1}} \\ &= \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta}P_{F,t}^{\varepsilon_{F,t}}A_t\left(\frac{1}{1-n}\right)^{-1}P_{F,t}^{-\varepsilon_{F,t}} \\ &= \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta}A_t(1-n) \end{aligned}$$

where we used

$$P_{F,t} = \left[\left(\frac{1}{1-n}\right)\int_n^1 P_{F,t}(i)^{1-\varepsilon_{F,t}}di\right]^{\frac{1}{1-\varepsilon_{F,t}}}$$

Now letting $n \to 0$ we finally get

$$IM_t = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta}A_t \tag{202}$$

Note that aggregate demand for the foreign economy in equilibrium is given by

$$Y_{F,t}(i) = A^*_{F,t}(i) + A_{F,t}(i)$$

and that clearly

$$\lim_{n\to 0} A_{F,t}(i) = 0$$

The demand of a small open economy is thus negligible to the rest of the world, as expected.

7.6 Prices

We will only focus on the derivation for domestic prices. The case of foreign prices is treated identically. A domestic firm i maximises the present value of net profits. From the process defined in equation (31) we get the following maximisation problem:

$$\max_{\mathcal{P},\mathcal{P}^*} \mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \theta_H^{t-\tau} \Delta_{t,\tau} \left(\mathcal{G}_{t|\tau}(i) - \mathcal{C}(Y^d_{H,t|\tau}) + \frac{(1-\theta_H)}{\theta_H} V_{t+1|\tau}(\cdot) \right) \right]$$

where $\mathcal{G}_{t|\tau}(i)$ are firm's i gross profits at time t, given by

$$\mathcal{G}_{t|\tau}(i) = A_{H,t|\tau}(i) \left(\frac{P_{H,t-1|\tau}}{P_{H,\tau-1}} \right)^{\gamma_h} \mathcal{P} + \xi_t A^*_{H,t|\tau}(i) \left(\frac{P^*_{H,t-1|\tau}}{P_{H,\tau-1}} \right)^{\gamma^*_h} \mathcal{P}^*$$

and $\mathcal{C}(\cdot)$ is the cost function and $V_{t+1}(\cdot)$ is the valuation function in the case where the firm gets to optimise the price at time $t+1$. The index $t \mid \tau$ indicates that values at time t are conditional on that the last price reset was at time τ. We interpret $\frac{P_{H,t|\tau}}{P_{H,\tau-1}} = 1$ for $t = \tau$. The price choice at time τ for domestically sold goods is $\mathcal{P}$ and $\mathcal{P}^*$ for goods sold abroad. Now, since the valuation function $V_{t+1|\tau}(\cdot)$ is independent of $\mathcal{P}$ and $\mathcal{P}^*$, we have

$$\frac{\partial V_{t+1|\tau}(\cdot)}{\partial X} = 0$$

for $X \in \{\mathcal{P}, \mathcal{P}^*\}$. Without loss of generality we assume $\tau = 0$. We will often suppress the conditionality on the last reset time for clarity. The first order condition for our maximisation problem with respect to $\mathcal{P}$ thus becomes

$$\begin{aligned} 0 = {} & \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \theta^t \Delta_{t,0} \left(\frac{\partial A_{H,t}(i)}{\partial \mathcal{P}} \left(\frac{P_{H,t-1}}{P_{H,-1}} \right)^{\gamma_h} \mathcal{P} + A_{H,t}(i) \left(\frac{P_{H,t-1}}{P_{H,-1}} \right)^{\gamma_h} \right) \right] \\ & - \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \theta^t \Delta_{t,0} MC_{H,t} \frac{\partial Y^d_{H,t}}{\partial \mathcal{P}} \right] \end{aligned}$$

By assuming that investment, maintenance and government consumption are dictated by the Dixit-Stiglitz framework, are priced in terms of consumption goods, and have the same elasticity of substitution, we can write:

$$A_{H,t}(i) = \bar{\alpha} \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\epsilon_{H,t}} \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} A_{H,t}$$

Since the last reset time of prices was at period 0, and taking into account the evolution of prices, we can write

$$A_{H,t}(i) = \bar{\alpha} P_0^{\epsilon_{H,t}\gamma_h} \mathcal{P}^{-\epsilon_{H,t}} \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} A_{H,t}$$

It is easy to see that we have

$$\frac{\partial A_{H,t}(i)}{\partial \mathcal{P}} \frac{\mathcal{P}}{-\epsilon_{H,t}} = A_{H,t}(i)$$

Further, we have from equation (42) that

$$\frac{\partial Y^d_{H,t}}{\partial \mathcal{P}} = \frac{\partial A_{H,t}(i)}{\partial \mathcal{P}}$$

Inserting the last two equations into the first order condition, we get

$$0 = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \theta^t \Delta_{t,0} A_{H,t} \left((1-\epsilon_{H,t}) \left(\frac{P_{H,t-1}}{P_{H,-1}} \right)^{\gamma_h} + \epsilon_{H,t} \mathcal{P}^{-1} MC_t \right) \right]$$

or equivalently with full notation

$$0 = \mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \theta^{t-\tau} \Delta_{t|\tau,\tau} A_{H,t|\tau} (1-\epsilon_{H,t}) \left(\mathcal{P} \left(\frac{P_{H,t-1|\tau}}{P_{H,\tau-1}} \right)^{\gamma_h} - \mathcal{M}_{H,t} MC_{t|\tau} \right) \right]$$

where $\mathcal{M}_{H,t} = \frac{\epsilon_{H,t}}{\epsilon_{H,t}-1}$.

Now we turn to deriving the law of motion for the price index. Using the law of the unconscious statistician, and the fact that marginal costs are the same for all firms, which implies that all firms optimise at the same price, we get:

$$E[P_{H,t+1}^{1-\epsilon_{H,t+1}}(i)|\check{I}_{t+1}] = \theta_H \left(\left(\frac{P_{H,t}}{P_{H,t-1}} \right)^{\gamma_H} P_{H,t}(i) \right)^{1-\epsilon_{H,t+1}} + (1-\theta_H) \mathcal{P}_{H,t+1}^{1-\epsilon_{H,t+1}}(i)$$

where the information set, $\check{I}_{t+1}$, includes everything known at time $t+1$ less the draw of firms allowed to optimise their prices. Since marginal cost of each firm is identical, the price choice is identical. Thus, we can write $\mathcal{P}_{H,t+1}(i) = \mathcal{P}_{H,t+1}$. It is clear that which firms are chosen to re-optimise their prices is irrelevant in the aggregate, i.e. $P_{H,t+1}^{1-\epsilon_{H,t+1}} = E[P_{H,t+1}^{1-\epsilon_{H,t+1}}|\check{I}_{t+1}]$. From the definition of price aggregation, given by equation (10), we can write

$$P_{H,t+1} = \left(\frac{1}{n} \int_0^n E_t[P_{H,t+1}^{1-\epsilon_{H,t+1}}(i)] di \right)^{\frac{1}{1-\epsilon_{H,t+1}}}$$

which yields

$$P_{H,t+1}^{1-\epsilon_{H,t+1}} = \frac{1}{n} \int_0^n \left[\theta_H \left(\left(\frac{P_{H,t}}{P_{H,t-1}} \right)^{\gamma_H} P_{H,t}(i) \right)^{1-\epsilon_{H,t+1}} + (1-\theta_H) \mathcal{P}_{H,t+1}^{1-\epsilon_{H,t+1}} \right] di$$

The integral of the latter factor is clearly just $n(1-\theta_H)\mathcal{P}_{H,t+1}^{1-\epsilon_{H,t+1}}$. The former can be written as

$$\theta_H \left(\frac{P_{H,t}}{P_{H,t-1}} \right)^{\gamma_H(1-\epsilon_{H,t+1})} \int_0^n P_{H,t}^{1-\epsilon_{H,t+1}}(i) di$$

By the definition of the price index, we have that

$$nP_{H,t}^{1-\epsilon_{H,t}} = \int_0^n P_{H,t}^{1-\epsilon_{H,t}}(i) di$$

which implies that

$$P_{H,t+1}^{1-\epsilon_{H,t}} = \theta_H \left(\left(\frac{P_{H,t}}{P_{H,t-1}} \right)^{\gamma_H} P_{H,t} \right)^{1-\epsilon_{H,t}} + (1-\theta_H) \mathcal{P}_{H,t+1}^{1-\epsilon_{H,t}} \tag{203}$$

This is the law of motion for domestic prices. The foreign case is handled identically.

7.7 Wages

Recall that the evolution of wages is given by

$$W_{t+1}(j) = \begin{cases} \mathcal{W}_{t+1}(j) & \Big| \text{with probability } (1-\theta_W) \\ \left(\frac{P_t}{P_{t-1}}\right)^{\gamma_W} \frac{Z_{t+1}}{Z_t} W_t(j) & \Big| \text{with probability } \theta_W \end{cases}$$

and that the wages chosen by a household at time τ is denoted $\mathcal{W}_\tau(j)$. By symmetry we can safely omit the reference to the household, and we simply write $\mathcal{W}_\tau$. The maximisation problem takes the form

$$\max_{\mathcal{W}} \mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \theta_W^{t-\tau} \beta^{t-\tau} \mathcal{U}_t(C_{t|\tau}(j), C_{t-1|\tau}, N_{t|\tau}(j)) \right] \tag{204}$$

such that

$$N_{t|\tau}(j) = \left(\frac{W_{t|\tau}(j)}{W_t} \right)^{-\epsilon_{W,t}} N_t$$

presuming that the budget constraint holds with equality for all t. The conditional labour supply constraint can be written as

$$N_{t|\tau}(j) = \left[\left(\frac{P_{t-1}}{P_{\tau-1}} \right)^{\gamma_W} \frac{Z_t}{Z_\tau} \mathcal{W} \right]^{-\epsilon_{W,t}} W_t^{\epsilon_{W,t}} N_t \tag{205}$$

The partial derivative with respect to the wage choice is given by

$$\frac{\partial N_{t|\tau}(j)}{\partial \mathcal{W}} \frac{\mathcal{W}}{-\epsilon_{W,t}} = N_{t|\tau}(j) \tag{206}$$

Furthermore, using the budget constraint, we can write consumption as a function of wages. Taking the derivative of consumption with respect to the wage choice gives

$$\frac{\partial C_{t|\tau}(j)}{\partial \mathcal{W}} = \frac{\partial}{\partial \mathcal{W}} \left(\frac{1}{P_t} \left[g(\cdot) + W_{t|\tau}(j) N_{t|\tau}(j) \right] \right) \tag{207}$$

$$= \frac{1}{P_t} \frac{\partial}{\partial \mathcal{W}} \left(W_{t|\tau}(j) N_{t|\tau}(j) \right) \tag{208}$$

where g is a function of non-consumption and non-labour related variables from the budget constraint, and for which it is evident that $\frac{\partial g(\cdot)}{\partial \mathcal{W}} = 0$. We have

$$N_{t|\tau}(j) W_{t|\tau}(j) = \left[\left(\frac{P_{t-1}}{P_{\tau-1}} \right)^{\gamma_W} \frac{Z_t}{Z_\tau} \mathcal{W} \right]^{-\epsilon_{W_t}} W_t^{\epsilon_{W,t}} N_t \left[\left(\frac{P_{t-1}}{P_{\tau-1}} \right)^{\gamma_W} \frac{Z_t}{Z_\tau} \mathcal{W} \right]$$

and thus

$$\frac{\partial N_{t|\tau}(j) W_{t|\tau}(j)}{\partial \mathcal{W}} = -(\epsilon_{W,t} - 1) \left[\left(\frac{P_{t-1}}{P_{\tau-1}} \right)^{\gamma_W} \frac{Z_t}{Z_\tau} \right]^{-\epsilon_{W_t}+1} W_t^{\epsilon_{W,t}} N_t \mathcal{W}^{-\epsilon_{W,t}} \tag{209}$$

$$= -(\epsilon_{W,t} - 1) \left[\left(\frac{P_{t-1}}{P_{\tau-1}} \right)^{\gamma_W} \frac{Z_t}{Z_\tau} \mathcal{W} \right]^{-\epsilon_{W_t}+1} W_t^{\epsilon_{W,t}} N_t \mathcal{W}^{-1} \tag{210}$$

$$= -(\epsilon_{W,t} - 1) N_{t|\tau}(j) \left[\left(\frac{P_{t-1}}{P_{\tau-1}} \right)^{\gamma_W} \frac{Z_t}{Z_\tau} \right] \tag{211}$$

and therefore

$$\frac{\partial C_{t|\tau}(j)}{\partial \mathcal{W}} = -P_t^{-1}(\epsilon_{W,t}-1)N_{t|\tau}(j)\left[\left(\frac{P_{t-1}}{P_{\tau-1}}\right)^{\gamma_W}\frac{Z_t}{Z_\tau}\right] \tag{212}$$

At time t we write

$$\frac{\partial \mathcal{U}_{t|\tau}(j)}{\partial \mathcal{W}} = MU_{C,t|\tau}\frac{\partial C_{t|\tau}(j)}{\partial \mathcal{W}} + MU_{N,t|\tau}\frac{\partial N_{t|\tau}(j)}{\partial \mathcal{W}}$$

Then the first order condition for the maximisation problem becomes

$$\begin{aligned} 0 = {} & \mathbb{E}_\tau\left[\sum_{t=\tau}^{\infty}\theta_W^{t-\tau}\beta^{t-\tau}\left(-MU_{C,t|\tau}P_t^{-1}(\epsilon_{W,t}-1)N_{t|\tau}(j)\left[\left(\frac{P_{t-1}}{P_{\tau-1}}\right)^{\gamma_W}\frac{Z_t}{Z_\tau}\right]\right)\right] \\ & - \mathbb{E}_\tau\left[\sum_{t=0}^{\infty}\theta_W^{t-\tau}\beta^{t-\tau}\left(MU_{N,t|\tau}\epsilon_{W,t}\mathcal{W}^{-1}N_{t|\tau}(j)\right)\right] \end{aligned}$$

multiply by $-\mathcal{W}$ and rearrange to produce

$$\begin{aligned} 0 = {} & \mathbb{E}_\tau\left[\sum_{t=\tau}^{\infty}\theta_W^{t-\tau}\beta^{t-\tau}\left((\epsilon_{W,t}-1)N_{t|\tau}(j)MU_{C,t|\tau}\frac{\mathcal{W}}{P_t}\left[\left(\frac{P_{t-1}}{P_{\tau-1}}\right)^{\gamma_W}\frac{Z_t}{Z_\tau}\right]\right)\right] \\ & - \mathbb{E}_\tau\left[\sum_{t=0}^{\infty}\theta_W^{t-\tau}\beta^{t-\tau}(\epsilon_{W,t}-1)N_{t|\tau}(j)\mathcal{M}_{W,t}MRS_{t|\tau}(j)\right] \end{aligned}$$

where $\mathcal{M}_{W,t} = \frac{\epsilon_{W,t}}{\epsilon_{W,t}-1}$.

Now using the law of the unconscious statistician, in the same manner as for the price index in Appendix A, section 7.6, the wage index can be written as

$$W_t = \left[\theta_W\left(\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_W}\frac{Z_t}{Z_{t-1}}W_{t-1}\right)^{1-\epsilon_{W,t}} + (1-\theta_W)\mathcal{W}_t^{1-\epsilon_{W,t}}\right]^{\frac{1}{1-\epsilon_{W,t}}} \tag{213}$$

8 Appendix B

In this section we list the underlying time series used in estimating the model parameters, as well as linking them to their corresponding variable in DYNIMO. See the QMM handbook (Daníelsson et al., 2019) for more details on the data.

8.1 Data description

$\mathbf{C}_t$ Private consumption. *Source*: Statistics Iceland/CBI. *Unit*: Chain-volume measure. Millions of kronas at constant 2005 prices.

$\mathbf{N}_t$ Total hours. *Source*: CBI. *Unit*: Total hours worked.

$\mathbf{EX}_t$ Export volume of goods and services. *Source*: Statistics Iceland/CBI. *Unit*: Chain-volume measure. Millions of kronas at constant 2005 prices.

$\mathbf{EX}_{g,t}$ Export volume of goods and services, excluding aluminium, marine products, airplanes, and ships as well as exports of manufacturing services. Source: Statistics Iceland. Unit: Chain-volume measure. Millions of kronas at constant 2005 prices.

G_t Government consumption. Source: Statistics Iceland/CBI. Unit: Chain-volume measure. Millions of kronas at constant 2005 prices.

Y_t Gross domestic production. Source: Statistics Iceland/CBI. Unit: Chain-volume measure. Millions of kronas at constant 2005 prices.

I_t Fixed investment. Source: Statistics Iceland/CBI. Unit: Chain-volume measure. Millions of kronas at constant 2005 prices.

IM_t Import volume of goods and services. Source: Statistics Iceland/CBI. Unit: Chain-volume measure. Millions of kronas at constant 2005 prices.

$P_{Y,t}$ GDP price deflator. Source: Statistics Iceland/CBI. Unit: Index (normalised such that mean is unity in 2005).

$P_{F,t}$ Import price deflator. Source: Statistics Iceland/CBI. Unit: Index.

S_t Real exchange rate. Source: CBI. Unit: Fraction.[44].

R_t Central Bank of Iceland monetary policy rate measured in annual yields. Source: CBI. Unit: Fraction.

W_t Wages. Source: Statistics Iceland. Unit: Index (normalised such that mean is unity in 2005).

P_t^* Trade weighted average of consumer prices in Iceland's main trading partners. Source: IMF/CBI. Unit: Index (normalised such that mean is unity in 2005).

A_t^* Trade weighted real GDP levels in Iceland's main trading partners. Source: OECD/CBI. Unit: Index (normalised such that mean is unity in 2005).

R_t^* Trade weighted foreign 3 month Treasury Bill interest rates of Iceland's main trading partners. Source: OECD/CBI. Unit: Fraction.

8.2 Data to model variables transformations

Table 10: Data to model variables transformations. We denote the trend of a variable X_t with X_t^T or X^{eq}.

Description	Transform
Output	$\mathring{Y}_t = \ln\left(\frac{Y_t}{Y_{t-1}}\right) - \ln\left(\frac{Y_t^T}{Y_{t-1}^T}\right)$
Consumption	$\mathring{C}_t = \ln\left(\frac{C_t}{C_{t-1}}\right) - \ln\left(\frac{C_t^T}{C_{t-1}^T}\right)$
Investment	$\mathring{I}_t = \ln\left(\frac{I_t}{I_{t-1}}\right) - \ln\left(\frac{I_t^T}{I_{t-1}^T}\right)$
Government consumption	$\mathring{G}_t = \ln\left(\frac{G_t}{G_{t-1}}\right) - \ln\left(\frac{G_t^T}{G_{t-1}^T}\right)$
Imports	$\mathring{IM}_t = \ln\left(\frac{IM_t}{IM_{t-1}}\right) - \ln\left(\frac{IM_t^T}{IM_{t-1}^T}\right)$
Exports	$\mathring{EX}_t = \frac{EX_t}{Y_t}\frac{Y_t^T}{EX_t^T} - 1$
Generic Exports	$\mathring{EX}_{g,t} = \frac{EX_{g,t}}{EX_t}\frac{EX_t^T}{EX_{g,t}^T} - 1$

(Continued on next page)

[44]The variable for real exchange rate in QMM, REX, is defined as S_t^{-1}.

Table 10: (continued)

Description	Transform
Inflation	$\mathring{\pi}_t = \Pi_{P,t} - \frac{\Pi^{eq}}{4}$
GDP deflator	$\mathring{P}_{Y,t} = \ln(P_{Y,t}) - \ln(P^T_{Y,t})$
Import deflator	$\mathring{P}_{F,t_t} = \ln\left(\frac{P_{F,t}}{P_{F,t-1}}\right) - \ln\left(\frac{P^T_{F,t}}{P^T_{F,t-1}}\right)$
Real exchange rate	$\mathring{S}_t = \ln(S_t) - \ln(S^T_t)$
Interest rate	$\mathring{R}_t = \frac{R_t - R^{eq}}{4}$
Total hours	$\mathring{N}_t = \ln(N_t) - \ln(N^T_t)$
Foreign output	$\mathring{A}^*_t = \ln(A^*_t) - \ln((A^*)^T)$
Foreign inflation	$\mathring{\pi}^*_t = \Pi^*_{P,t} - \frac{(\Pi^*)^{eq}}{4}$
Foreign interest rates	$\mathring{R}^*_t = \frac{R^*_t - (R^*)^{eq}}{4}$

9 Appendix C

9.1 Steady state relations

In this appendix we derive steady state values required to calculate the percentage deviations from steady state but were not derived in the text. We start by stating the assumptions:

- $\Pi_P = \Pi_H = \Pi_F = \Pi^*_H = 1$
- $P_H = U = 1$
- $Z_i = 1$
- $B^*_H = NX = 0$

where Z_i is in the set of all shocks of that form in the model. In addition, we assume no adjustment cost in steady state. Let us first show that $R = R^* = \beta^{-1}$. Equation (143) states

$$1 = R_t E[\Lambda_{t,t+1}]$$

and we previously defined

$$\Lambda_{t,t+1} = \beta \frac{MU_{C,t+1}}{MU_{C,t}} \frac{P_t}{P_{t+1}}$$

From $\Pi_P = \Pi_P = \Pi_Z = 1$ we get that

$$\Lambda = \beta$$

and therefore

$$R = \beta^{-1}$$

Equation (146) gives the relationship between R^*_t and the domestic economy:

$$1 = R^*_t(1 - \Gamma_{B,t})\mathbb{E}[\Lambda_{t,t+1}\Pi_{\xi,t+1}]$$

By assumption $\Gamma_B = 0$ and $\Pi_\xi = 1$ follows from definitions and $T = 1$ from below. We have just shown that $\Lambda = \beta$, and we thus get

$$1 = R^* \beta$$

Next we establish the steady state relationships of output, export, import and the exchange rate. Note that since we assume $P_H = 1$, the definition of the price index in steady state, given by

$$1 = (1-\alpha) P_H^{1-\eta} + \alpha P_F^{1-\eta}$$

implies that $P_F = 1$. From equation (47), which states that

$$IM_t = \alpha \left(\frac{P_{F,t}}{P_t} \right)^{-\eta} A_t$$

we immediately get

$$\alpha IM = A$$

By definition we have

$$P_{Y,t} Y_t = P_{H,t} Y_{H,t}^h + \xi_t P_{H,t}^* EX_{H,t}$$

From equations (46) we know that we can write

$$Y_{H,t}^h = (1-\alpha) \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} A_t$$

From which we directly see

$$Y_H^h = (1-\alpha) A$$

Again using the definition of the price index and import demand we can write

$$\frac{P_{Y,t}}{P_t} Y_t = A_t + \xi_t \frac{P_{H,t}^*}{P_t} NX_t$$

In steady state, we assume that $NX = 0$ and it is clear that we can write

$$Y = P_Y^{-1} A$$

and

$$EX = T \cdot IM$$

We can thus write

$$T^\alpha A = (1-\alpha) P_H^{1-\eta} A + \alpha S P_H^* T P_F^{-\eta} A$$

Since $P_Y = T^{-\alpha}$. Using that $P_F = P_H = 1$ and the definition of terms of trade

$$T = \frac{P_F}{S P_{H,t}^*}$$

we get

$$T^\alpha = 1 - \alpha + \alpha$$

and thus $T = P_Y = 1$, which implies that

$$S P_{H,t}^* = 1, \quad Y = A, \quad \text{and } IM = EX$$

From the calculations above we can thus conclude that

$$\alpha Y = \alpha A = IM = EX$$

Next we show that $R^K = \beta^{-1} - 1 + \delta$. Equation (28) states

$$\begin{aligned} 1 = & Q_t Z_{I,t}\left(1 - \Gamma_{I,t}(\cdot) - \Gamma'_{I,t}(\cdot)\left(\frac{I_t}{I_{t-1}}\right)\right) \\ & + \mathbb{E}_t\left[Q_{t+1}\frac{P_{t+1}}{P_t}\Lambda_{t,t+1}Z_{I,t+1}\Gamma'_{I,t+1}(\cdot)\left(\frac{I_{t+1}}{I_t}\right)^2\right] \end{aligned}$$

Since we assume no adjustment costs in equilibrium we get $Q = 1$, but we know from section 5.3 that

$$Q = \frac{\beta}{(1-(1-\delta))\beta}R^K$$

Straightforwardly from these two equations we get

$$R^K = \beta^{-1} - 1 + \delta$$

We now give the steady state relationship for the domestic markup. We know from above that $EX = \alpha Y$ and, by assumption, $EX_E = \alpha_E EX$. Thus

$$K_E = Y_E = EX_E = \alpha \cdot \alpha_E Y$$

Since

$$Y = Y_g + Y_E$$

it follows that

$$Y_g = (1 - \alpha \cdot \alpha_E)Y$$

Therefore, from

$$K = K_g + K_E$$

we get

$$\begin{aligned} \frac{K}{Y} &= \frac{K_g}{Y} + \frac{K_E}{Y} \\ &= \frac{K_g}{Y_g}(1 - \alpha \cdot \alpha_E) + \alpha \cdot \alpha_E \\ &= \frac{MC\psi_H}{R^K}(1 - \alpha \cdot \alpha_E) + \alpha \cdot \alpha_E \end{aligned}$$

We have that $R^K = \beta^{-1} - 1 + \delta$ from above, and $MC = \mathcal{M}_H^{-1}$ from section 5.4.1. Thus

$$\frac{K}{Y} = \frac{(1-\alpha \cdot \alpha_E)\psi_H}{\mathcal{M}_H(\beta^{-1} - 1 + \delta)} + \alpha \cdot \alpha_E \tag{214}$$

Isolating M_H in the last equation gives

$$\mathcal{M}_H = \frac{(1-\alpha \cdot \alpha_E)\psi_H}{(\beta^{-1} - 1 + \delta)\left(\frac{Y}{K} - \alpha \cdot \alpha_E\right)} \tag{215}$$

From the stationary equation of the law of motion of capital, given in equation (65):

$$1 = (1-\delta)\overline{K}_{S,t}\overline{K}_{S,t+1}^{-1}\Pi_{Z,t}^{-1} + Z_{I,t}\left(1 - \Gamma_I\left(\frac{\overline{I}_t}{\overline{I}_{t-1}}\Pi_{Z,t}\right)\right)\overline{I}_t\overline{K}_{S,t+1}^{-1} \tag{216}$$

Inserting $\Pi_Z = Z_I = 0$ and $\Gamma_I(1) = 0$, in steady state we clearly have

$$1 = (1-\delta) + \frac{I_t}{K_S} \tag{217}$$

Since $U = 1$ in equilibrium, we get the result

$$\delta K = I \tag{218}$$

From definition we have $A = C+I+G+NX$. Since we assume $NX = 0$ and from the calculations above we have $A = Y$, therefore we get

$$1 = \frac{C}{Y} + \frac{I}{Y} + \frac{G}{Y} \tag{219}$$

10 Appendix D

10.1 Figures

Correspondence between model variables and estimation variables, frequently used in titles of figures, can be found in table 12.

10.1.1 Impulse response functions

Monetary policy shock (ε_R)

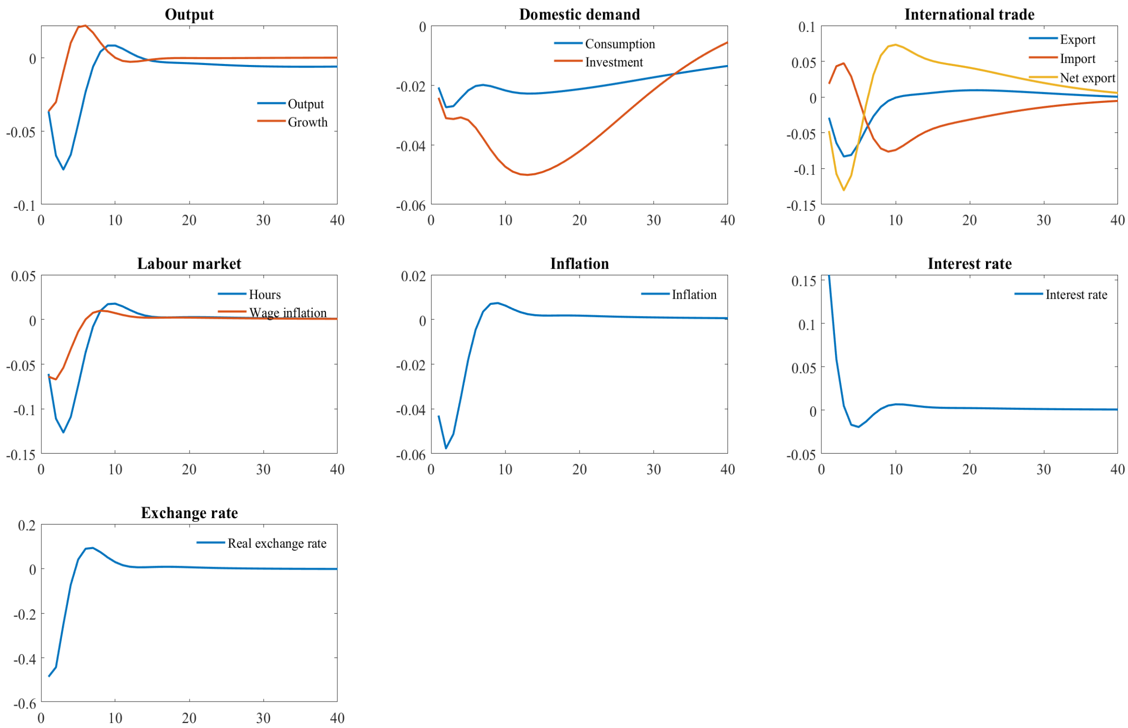

Figure 2: Responses, in percent, to a 1 standard deviation impulse of $\varepsilon_{R,0}$.

Labour augmenting technology shock (ε_H)

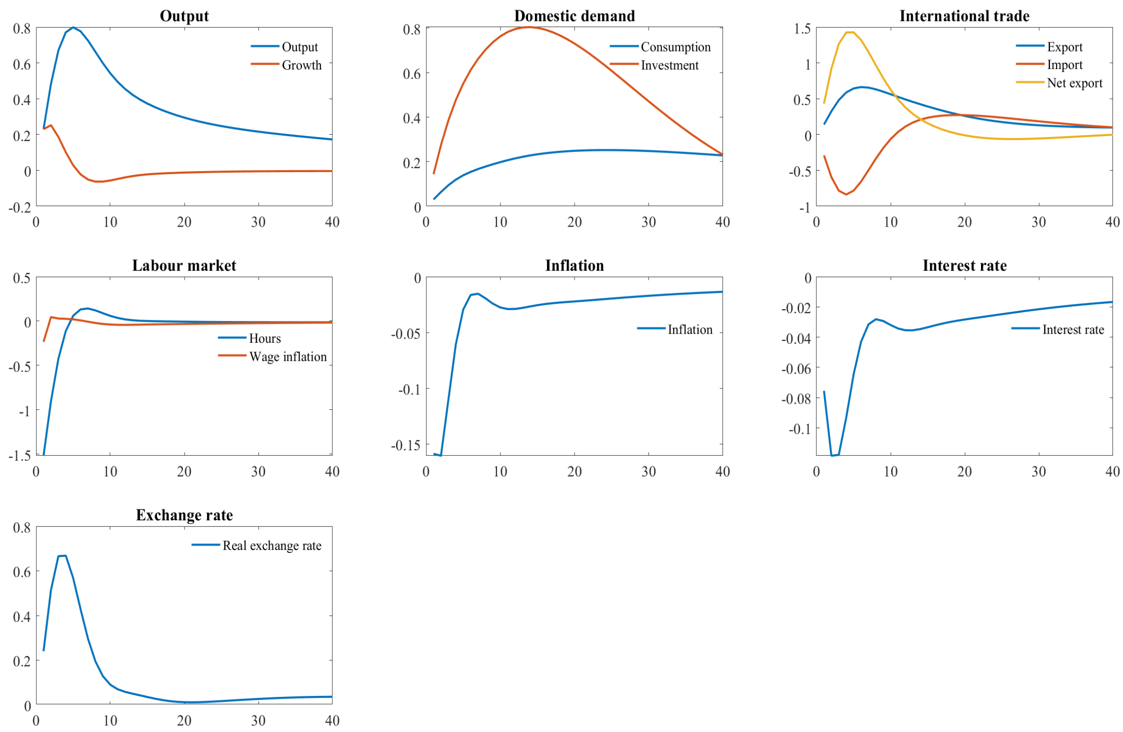

Figure 3: Responses, in percent, to a 1 standard deviation impulse of $\varepsilon_{H,0}$.

Domestic markup shock (ε_{μ_H})

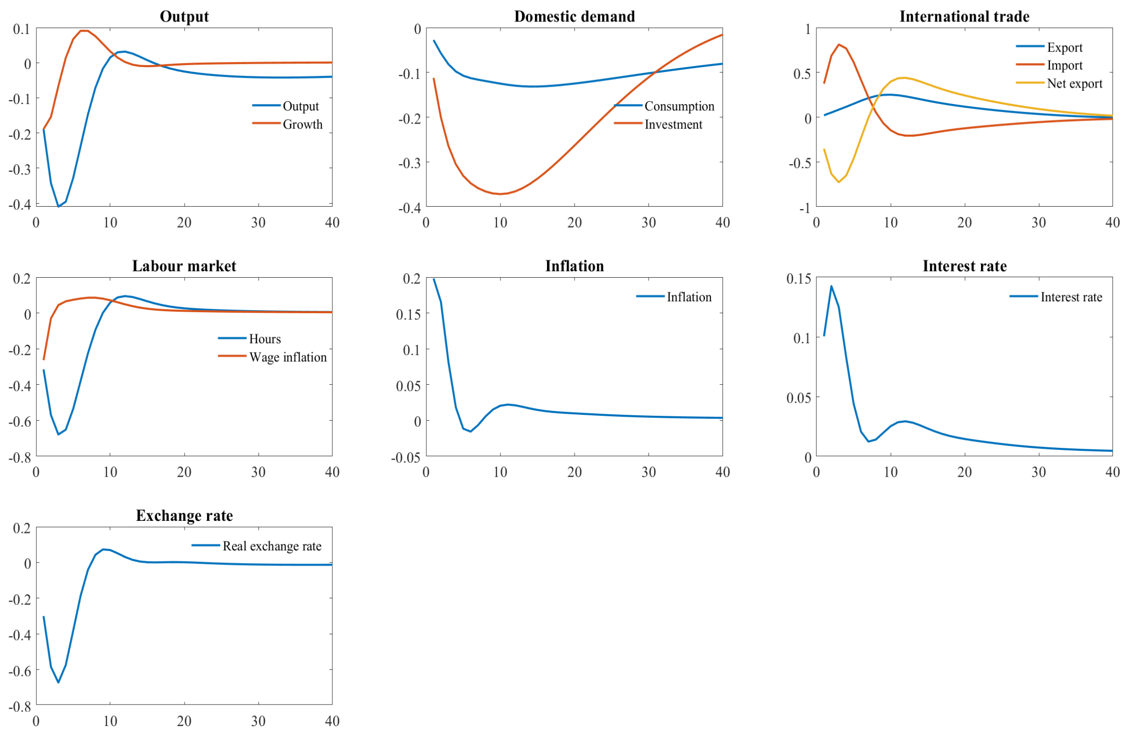

Figure 4: Responses, in percent, to a 1 standard deviation impulse of $\varepsilon_{\mu_H,0}$.

Export markup shock ($\varepsilon^*_{\mu_H}$)

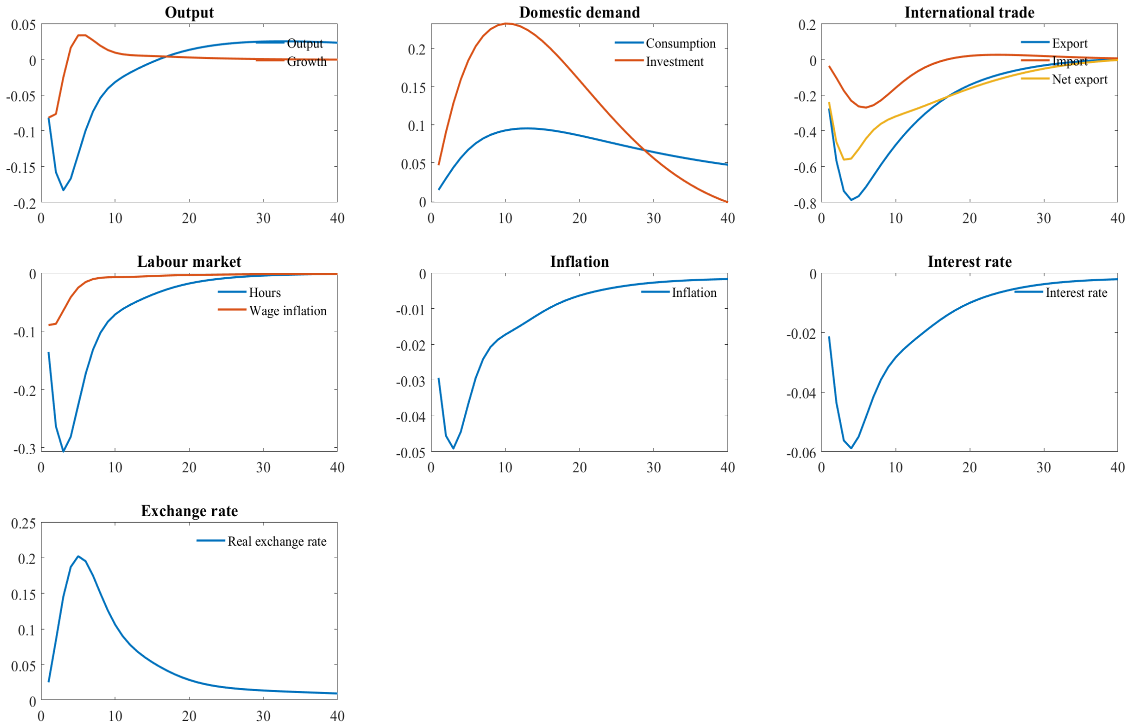

Figure 5: Responses, in percent, to a 1 standard deviation impulse of $\varepsilon^*_{\mu_H,0}$.

Wage markup shock (ε_{μ_W})

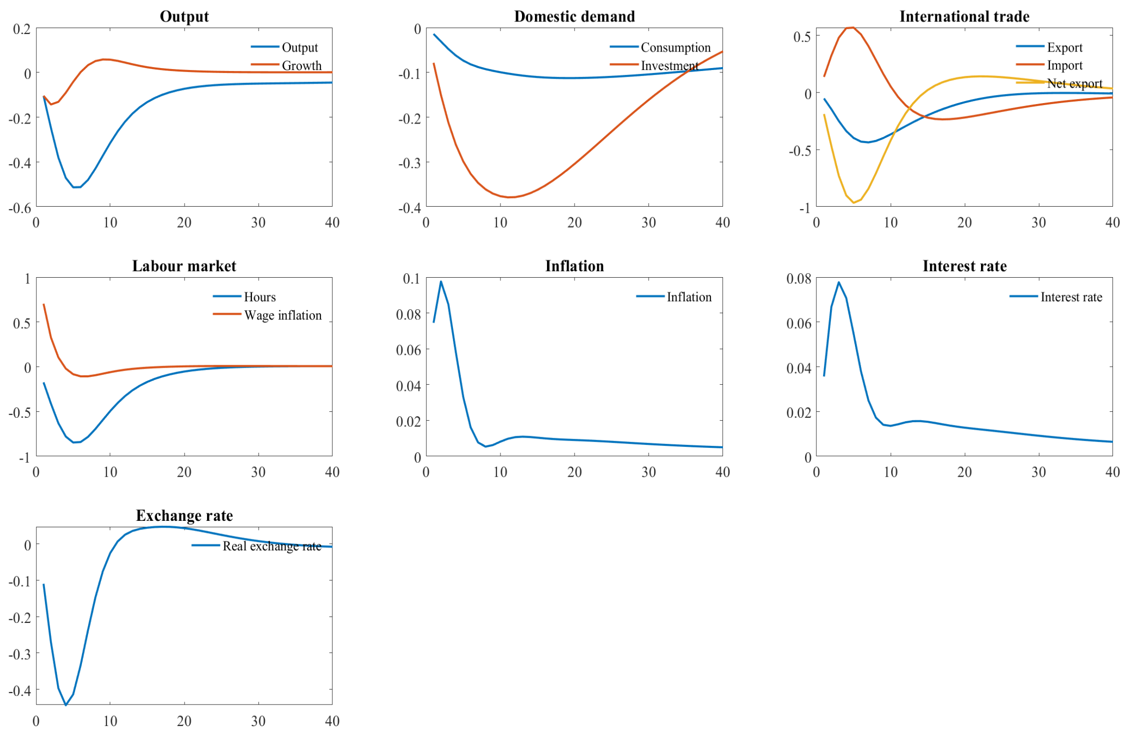

Figure 6: Responses, in percent, to a 1 standard deviation impulse of $\varepsilon_{\mu_W,0}$.

Risk premium shock (ε_B)

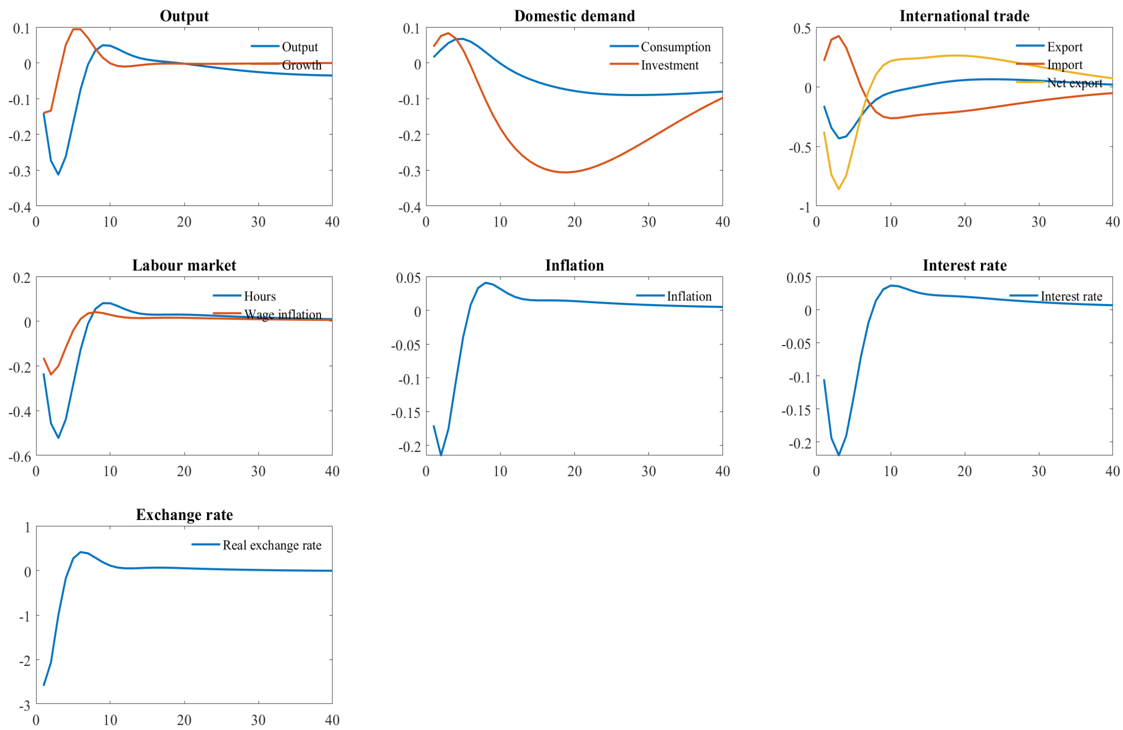

Figure 7: Responses, in percent, to a 1 standard deviation impulse of $\varepsilon_{B,0}$.

10.1.2 Forecast error variance

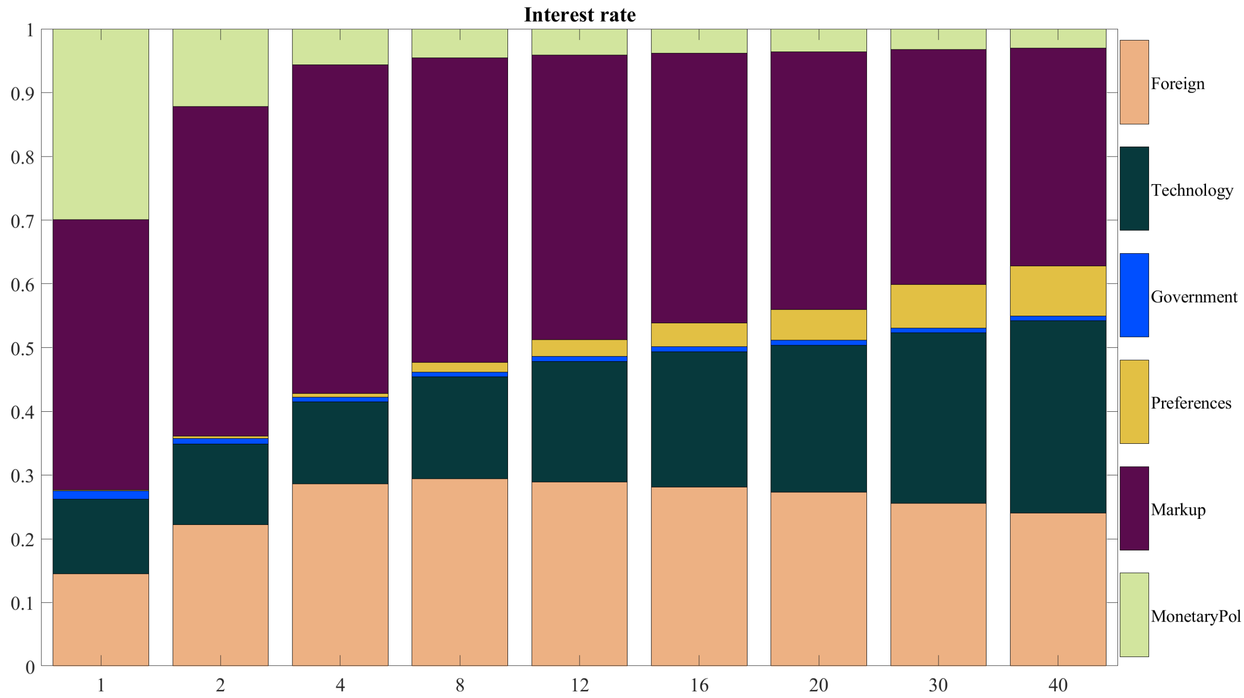

Figure 8: Forecast error variance decomposition for the interest rate.

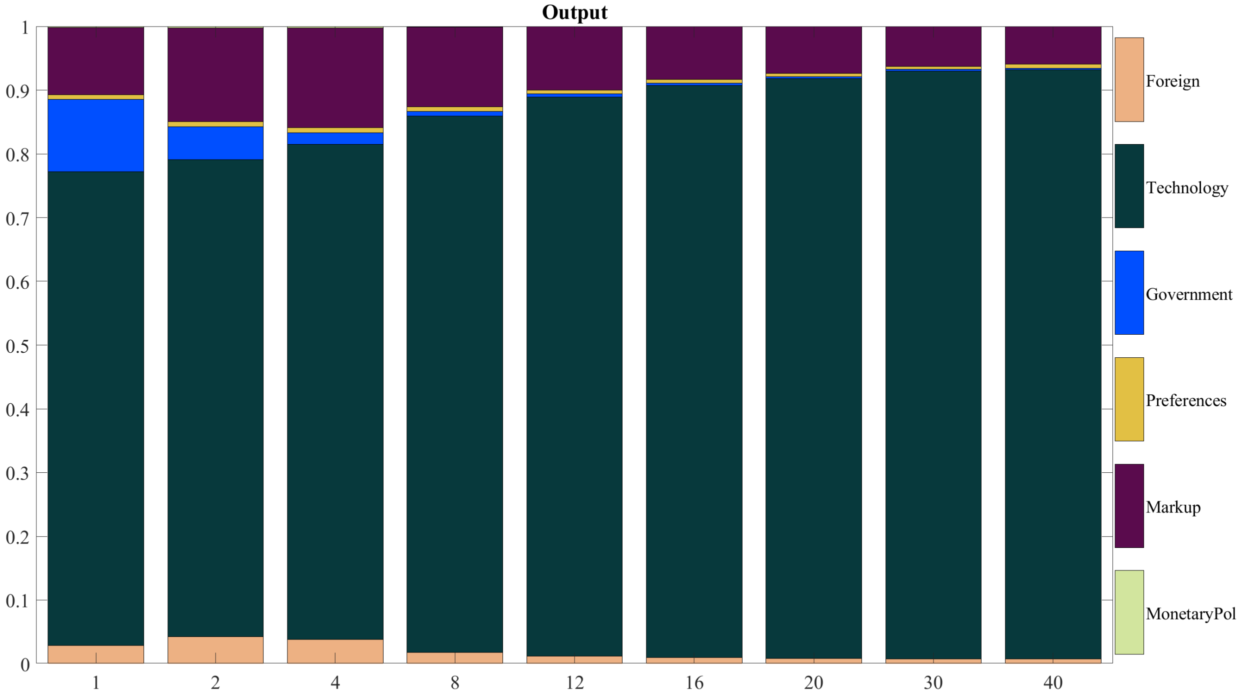

Figure 9: Forecast error variance decomposition for output.

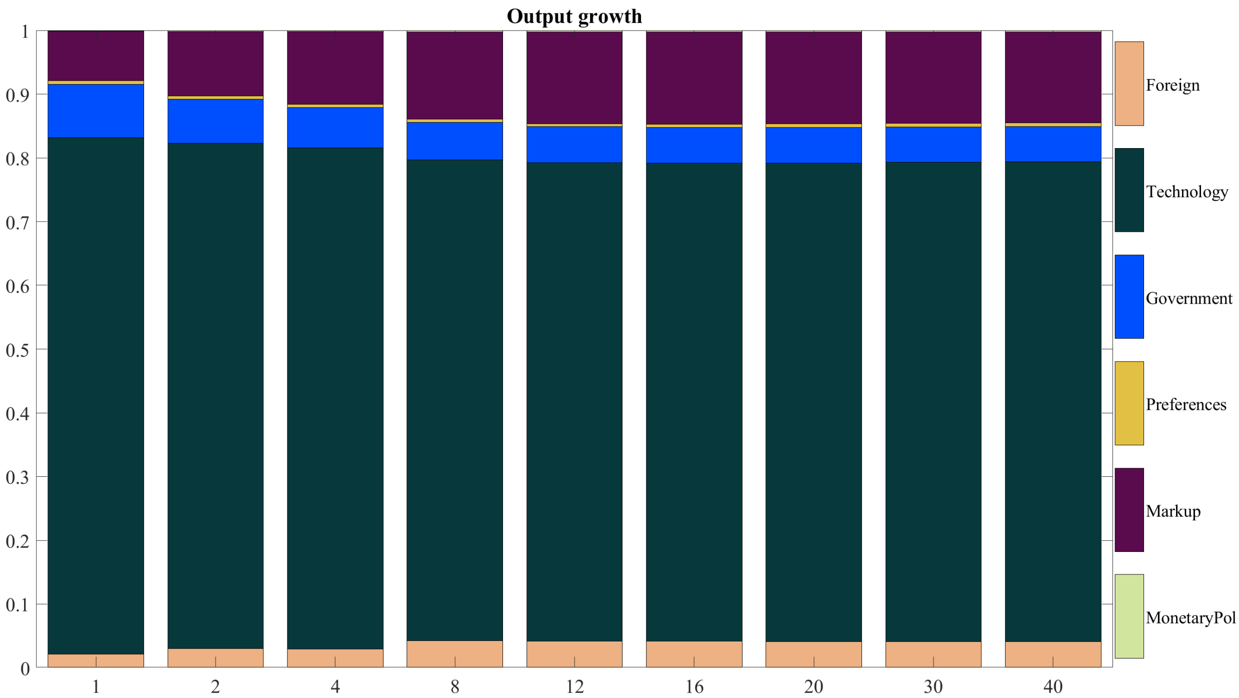

Figure 10: Forecast error variance decomposition for output growth.

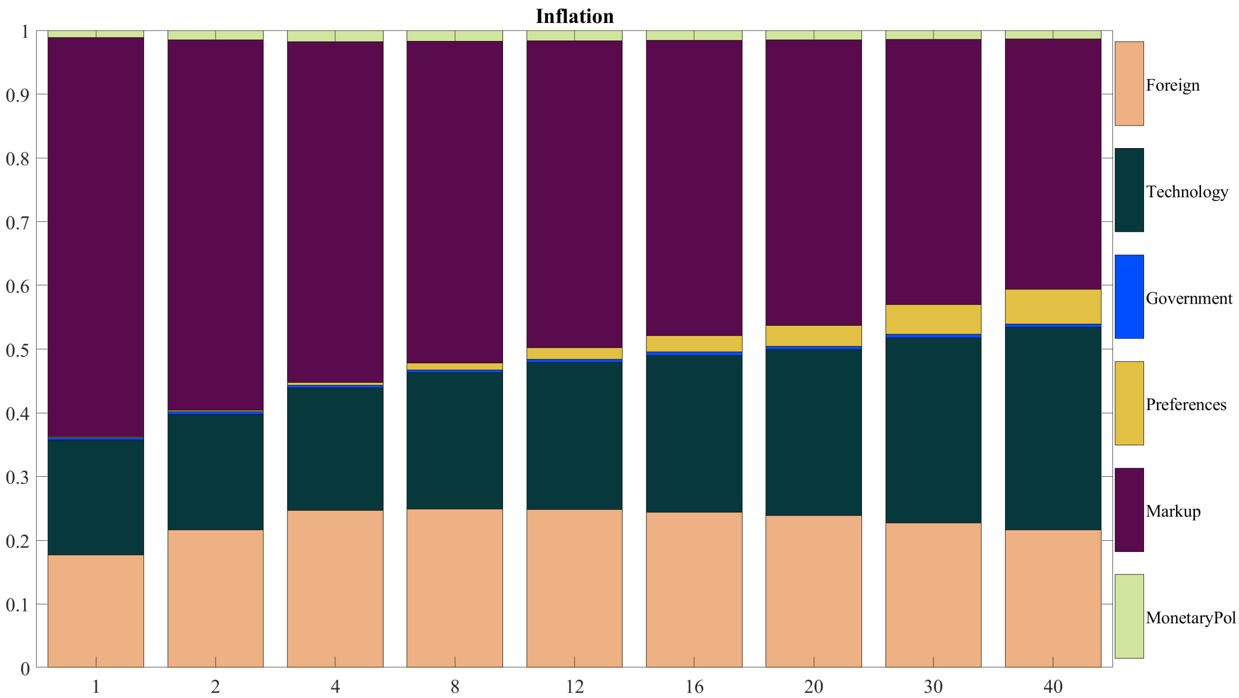

Figure 11: Forecast error variance decomposition for inflation.

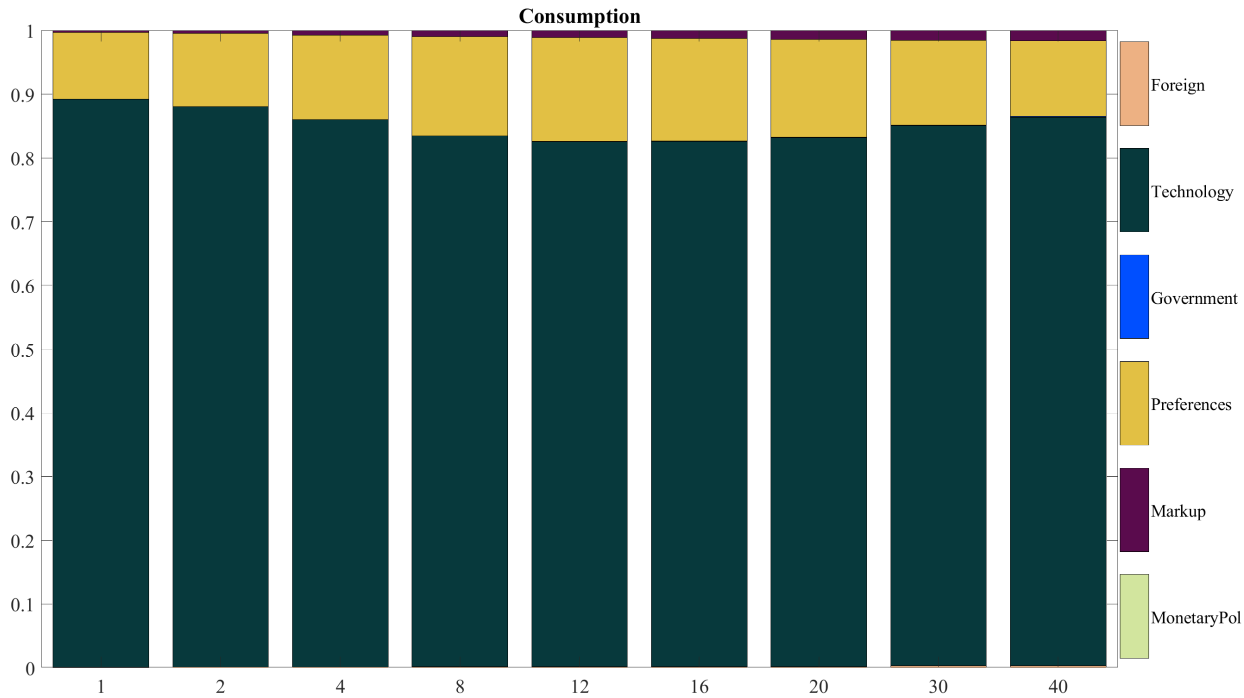

Figure 12: Forecast error variance decomposition for consumption.

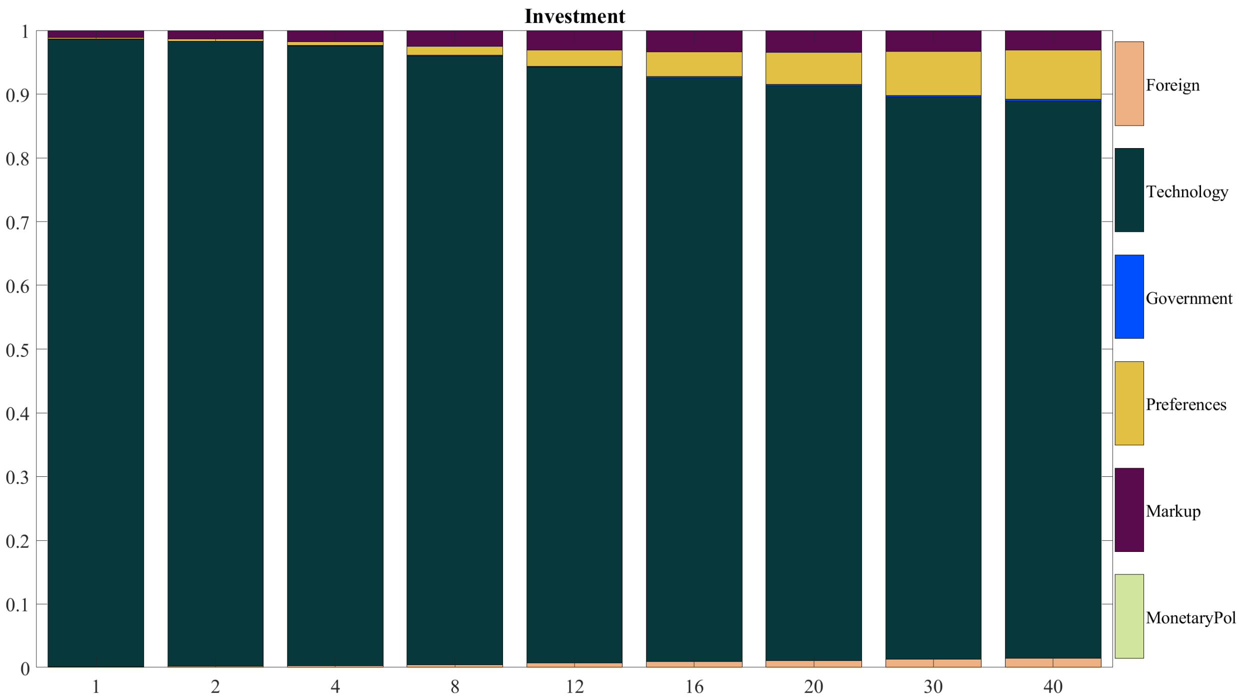

Figure 13: Forecast error variance decomposition for investment.

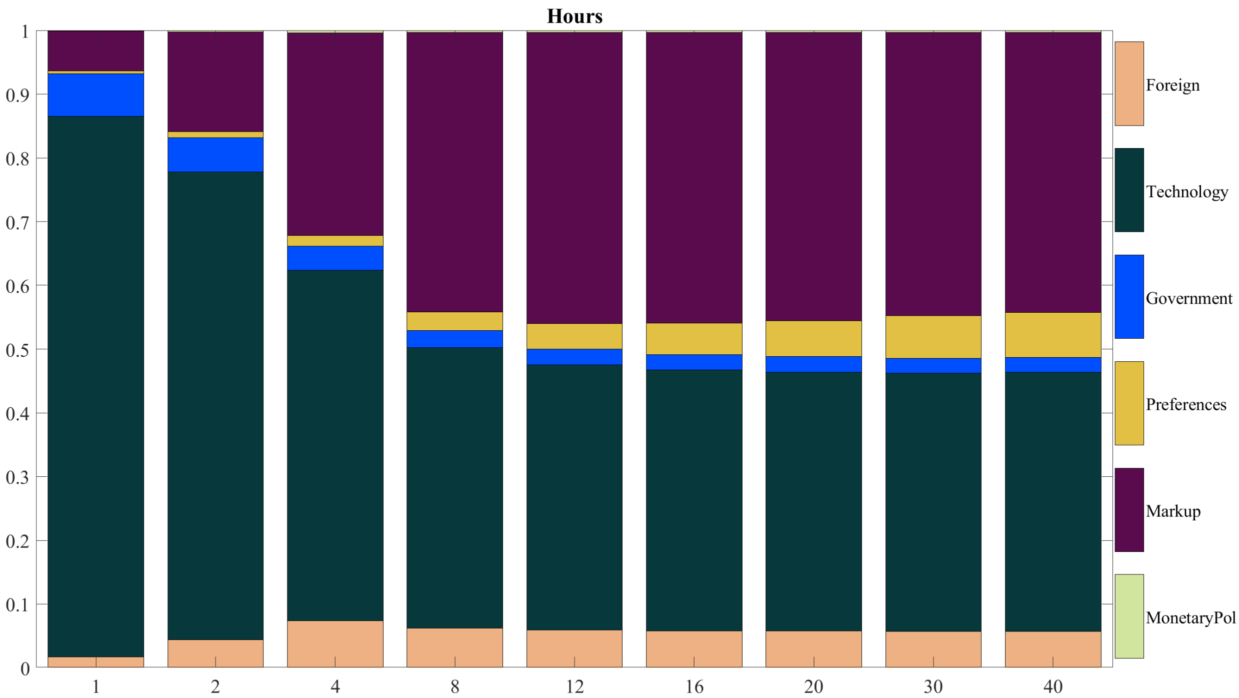

Figure 14: Forecast error variance decomposition for hours.

10.1.3 Posteriors and priors

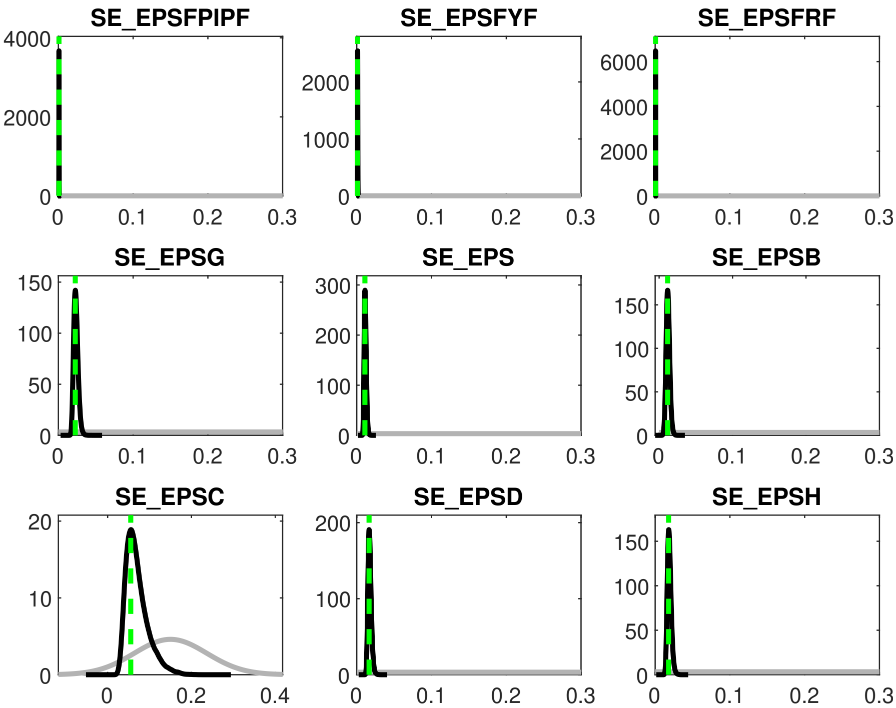

Figure 15: Unnormalised probability densities of priors and posteriors.

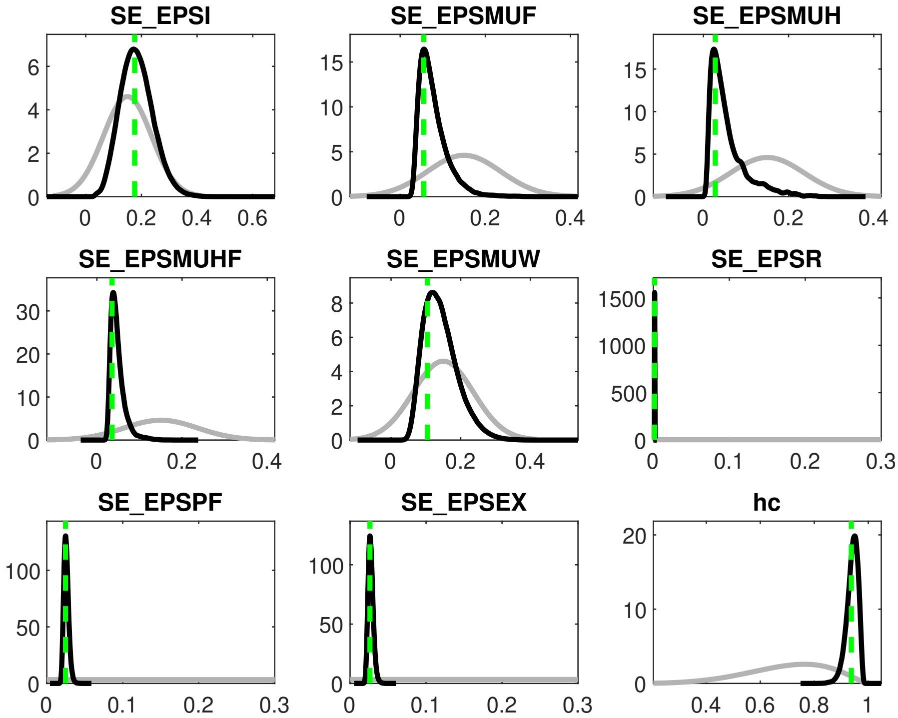

Figure 16: Unnormalised probability densities of priors and posteriors.

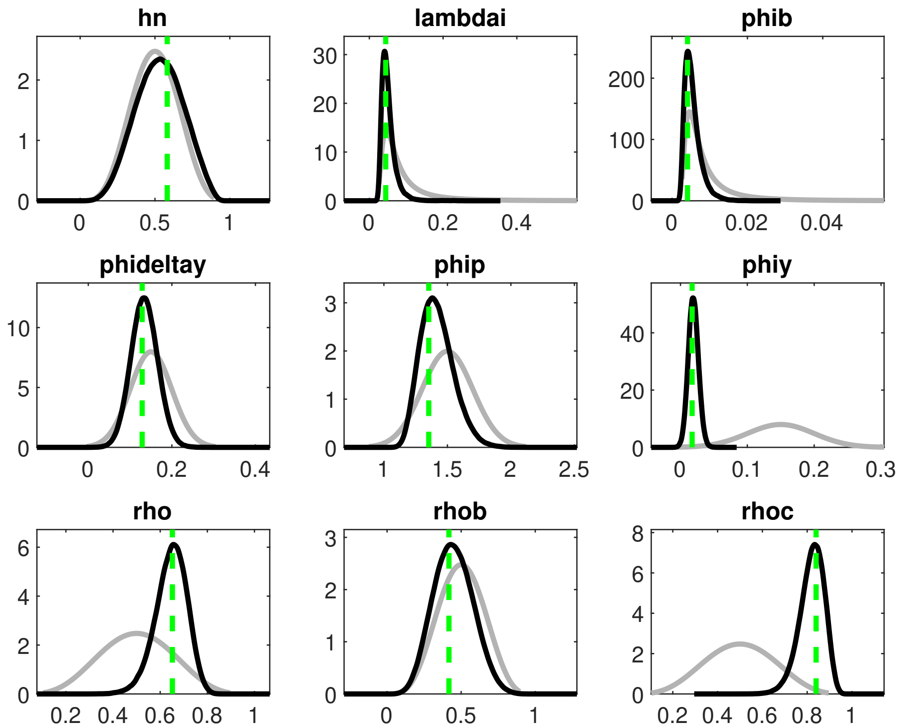

Figure 17: Unnormalised probability densities of priors and posteriors.

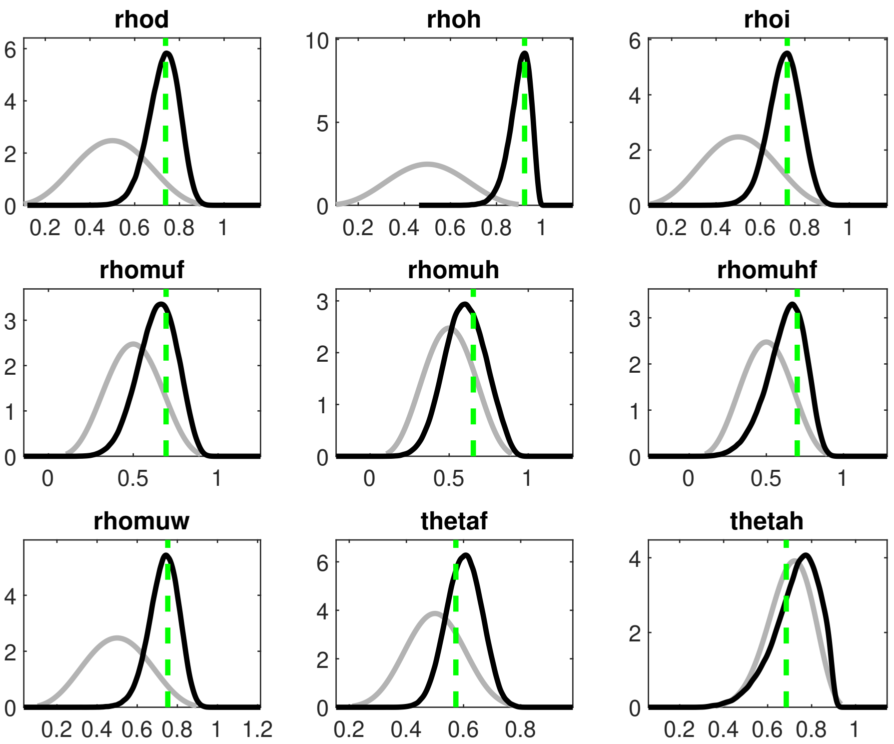

Figure 18: Unnormalised probability densities of priors and posteriors.

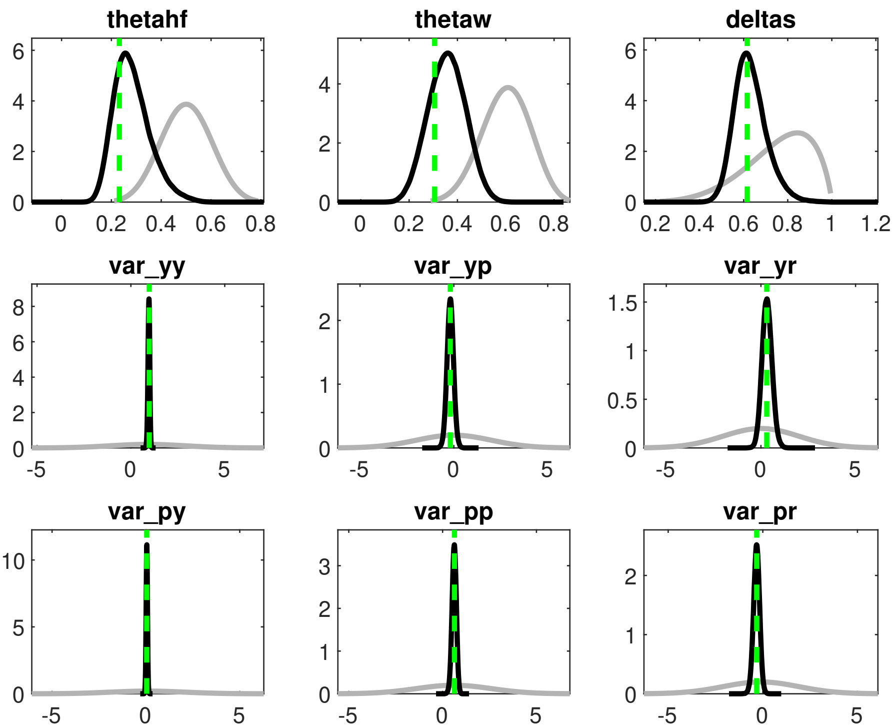

Figure 19: Unnormalised probability densities of priors and posteriors.

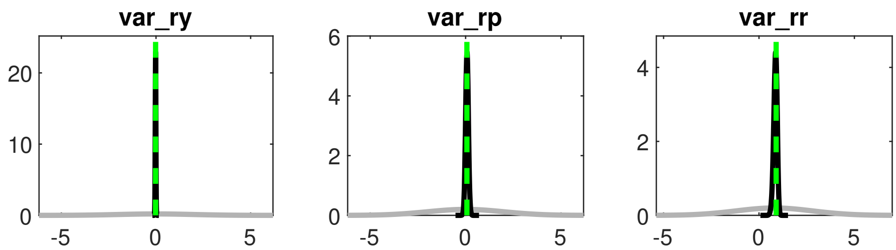

Figure 20: Unnormalised probability densities of priors and posteriors.

10.1.4 Non-grouped forecast error variance decomposition

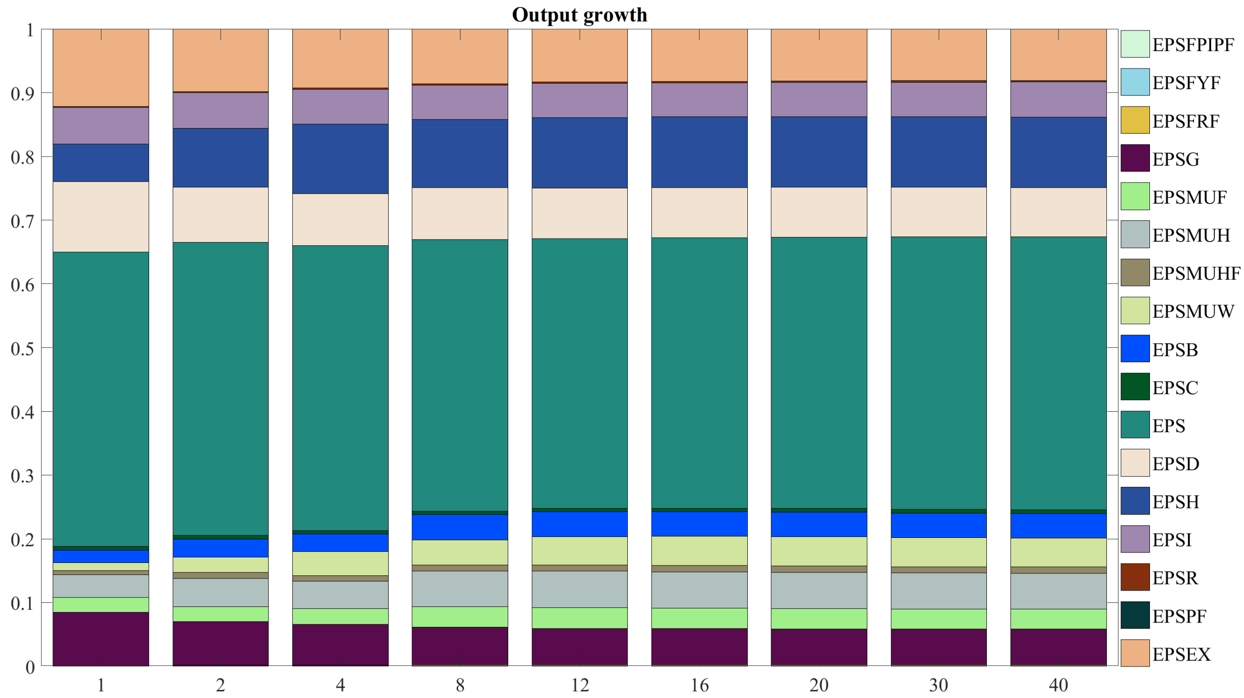

Figure 21: Forecast error variance decomposition for output growth.

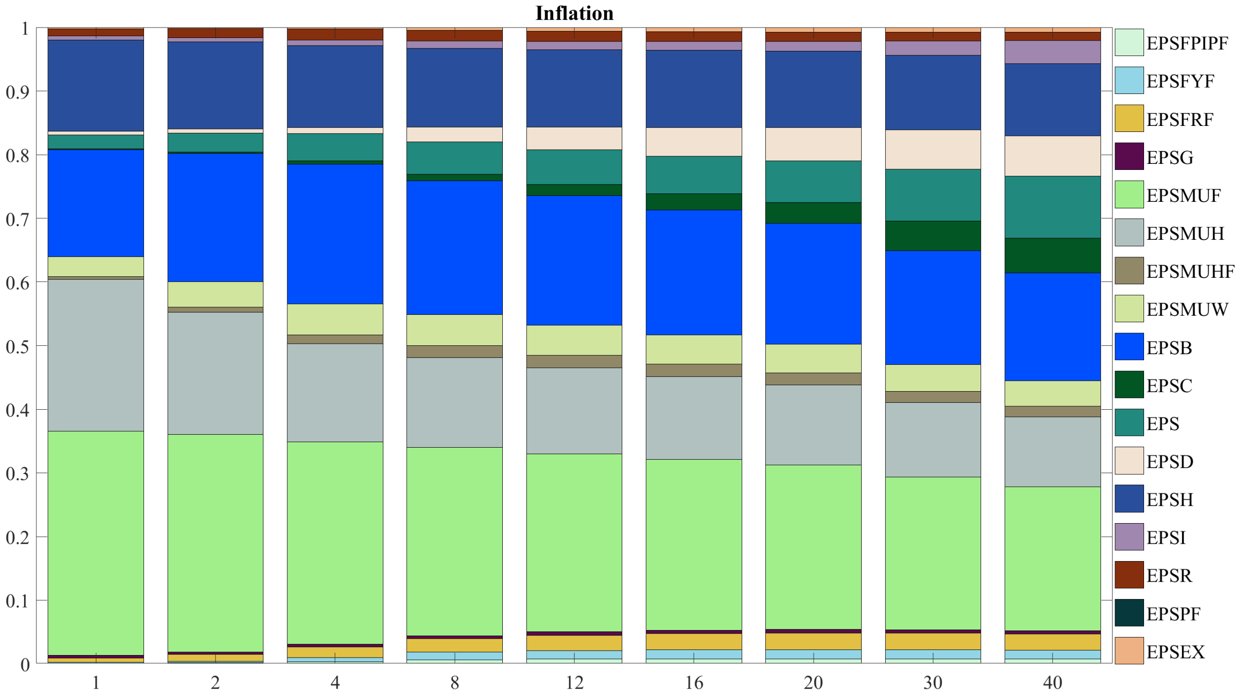

Figure 22: Forecast error variance decomposition for inflation.

10.1.5 Convergence diagnostics

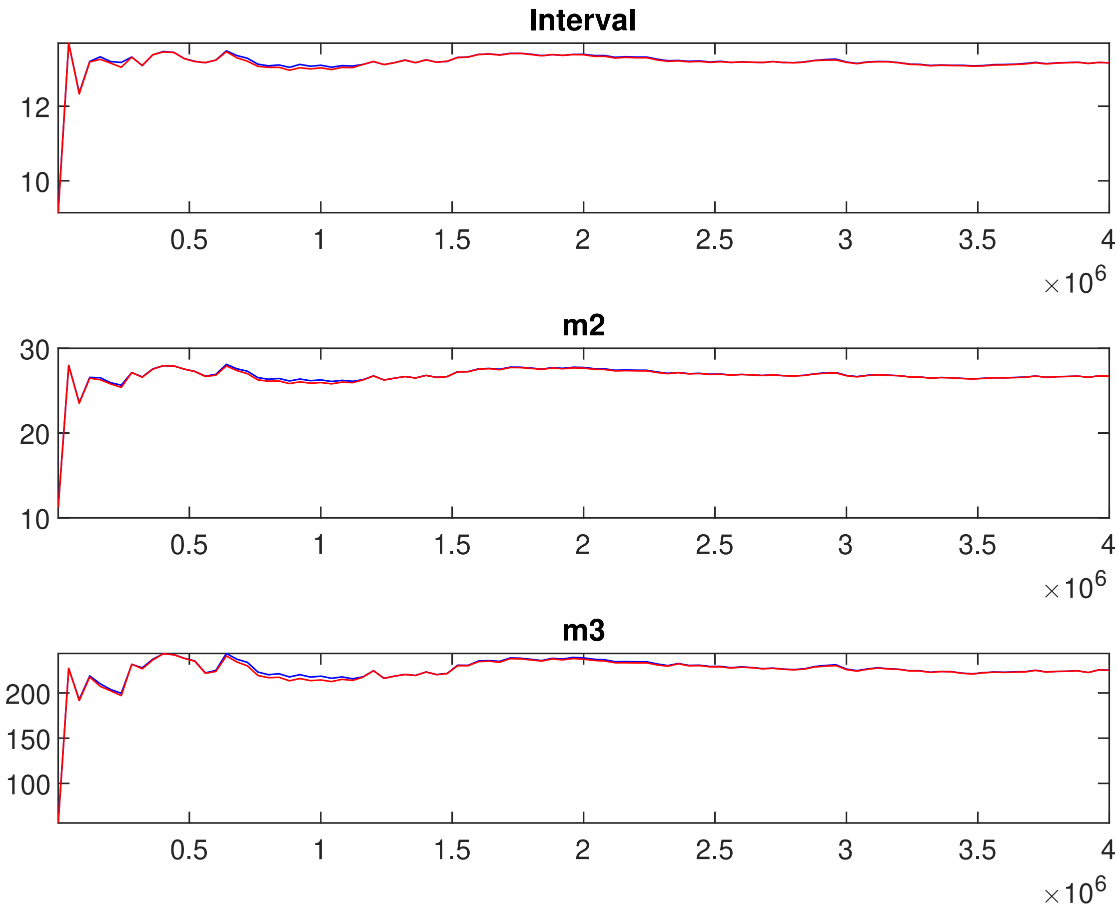

Figure 23: Multivariate convergence diagnostics for the Metropolis-Hastings. The first, second and third rows are respectively the criteria based on the eighty percent interval, the second and third moments. The different parameters are aggregated using the posterior kernel.

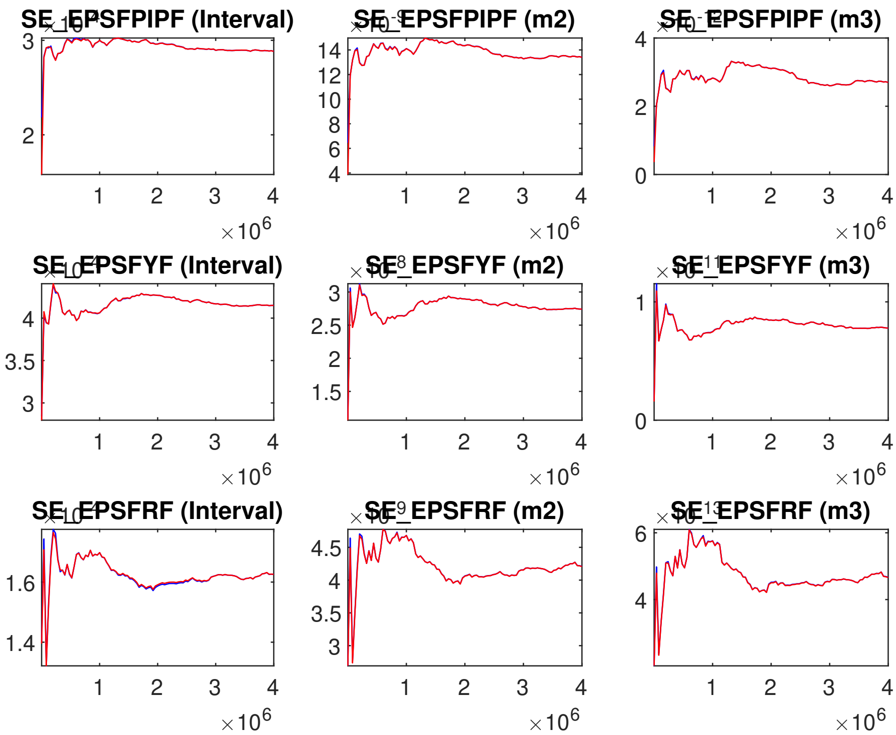

Figure 24: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

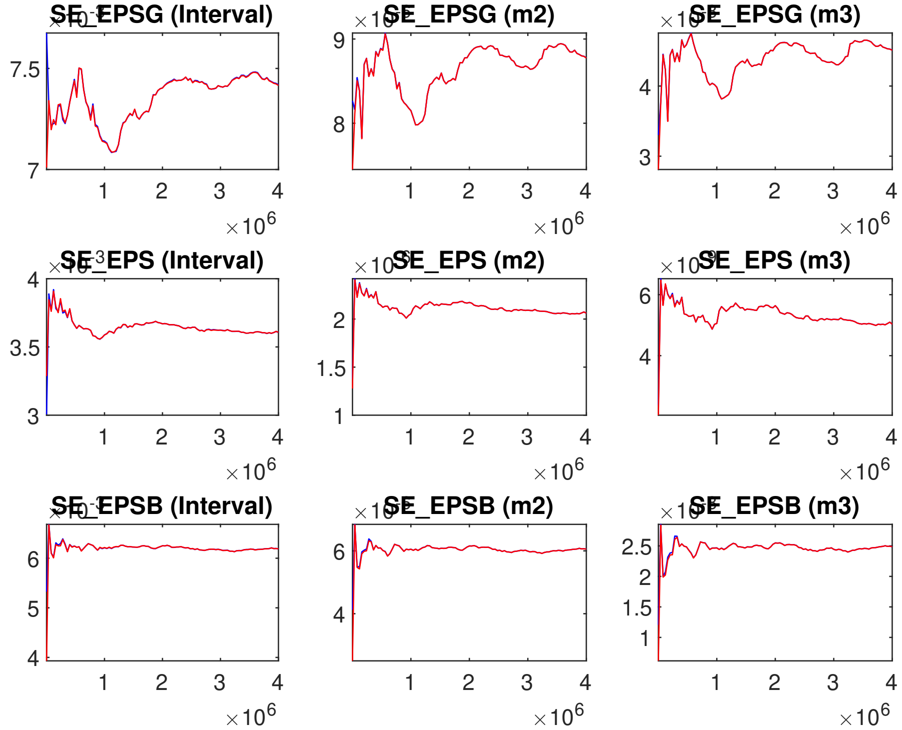

Figure 25: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

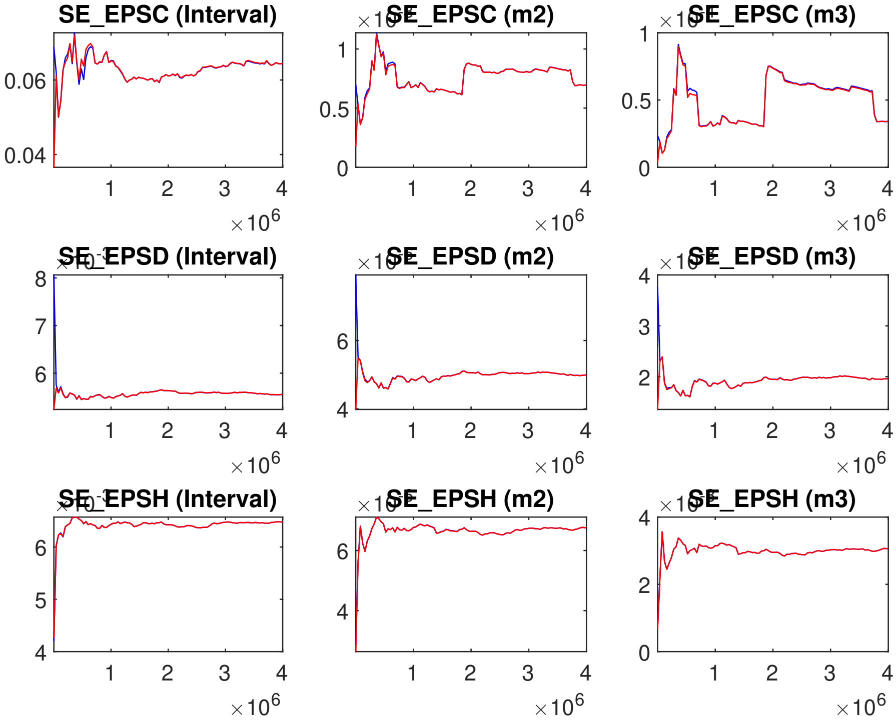

Figure 26: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

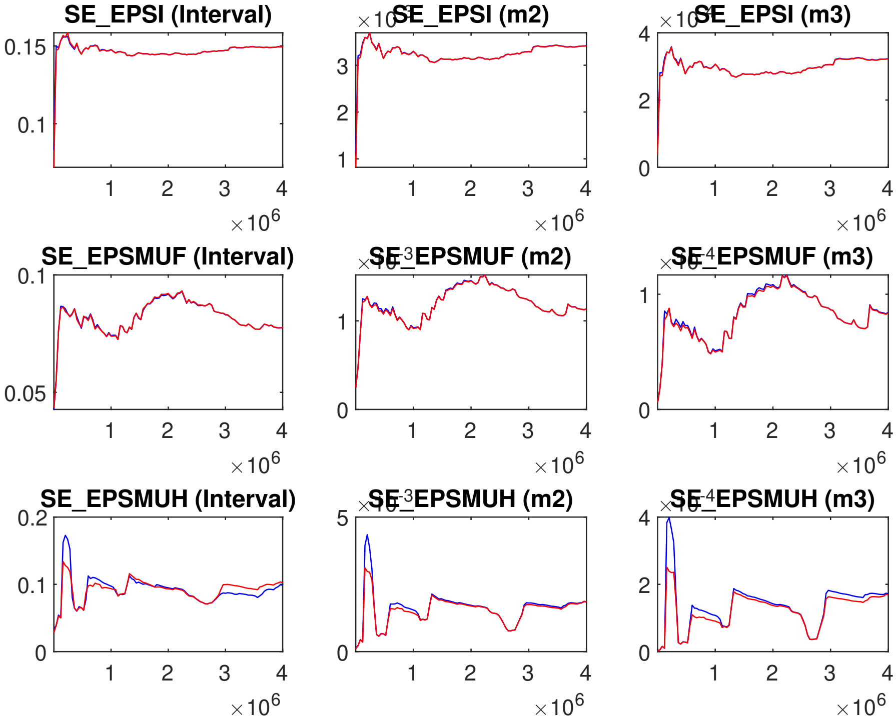

Figure 27: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

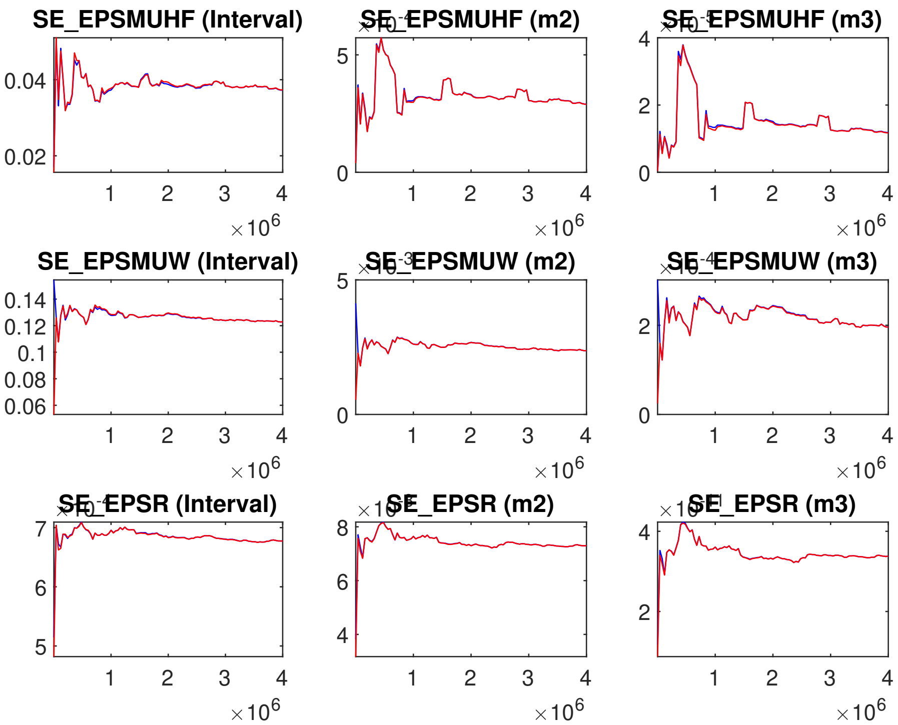

Figure 28: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

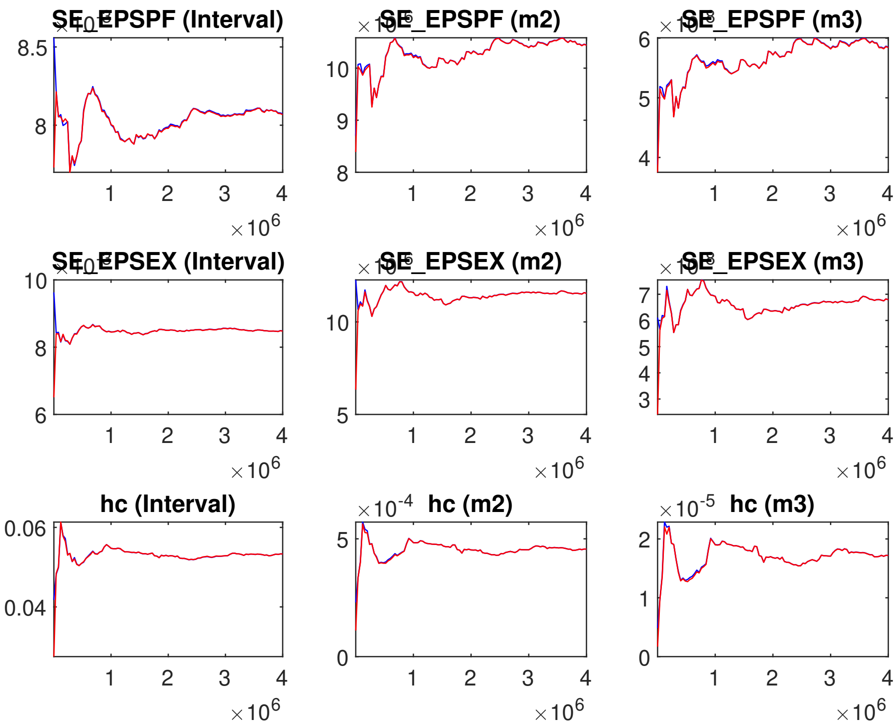

Figure 29: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

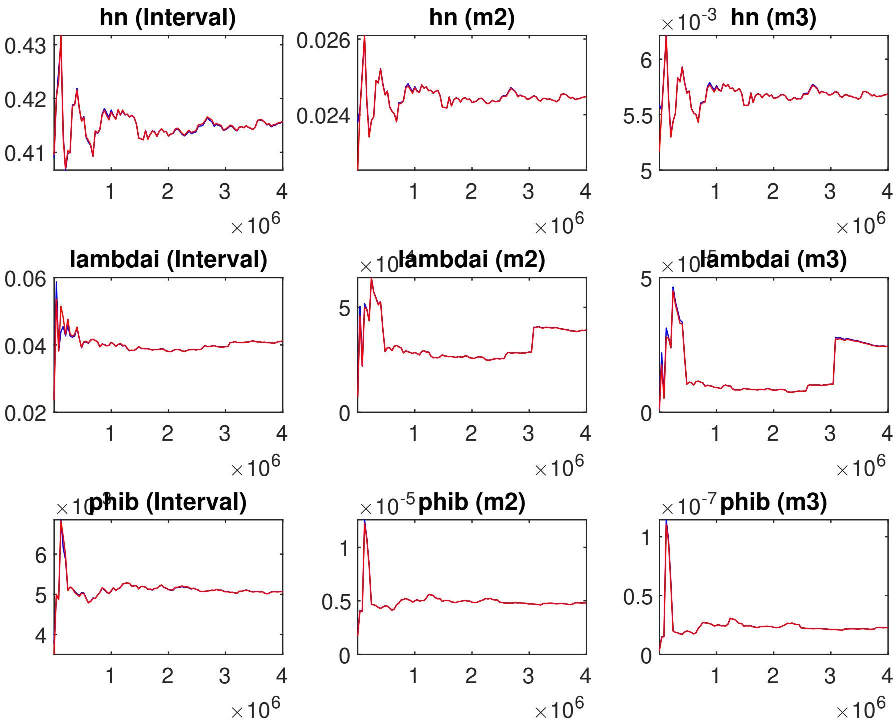

Figure 30: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

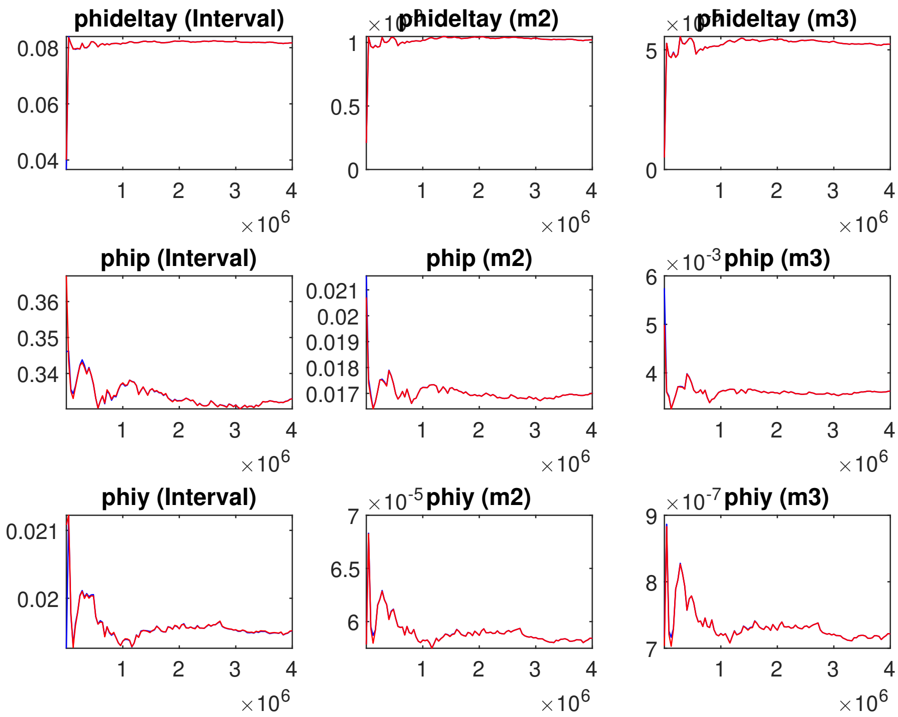

Figure 31: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

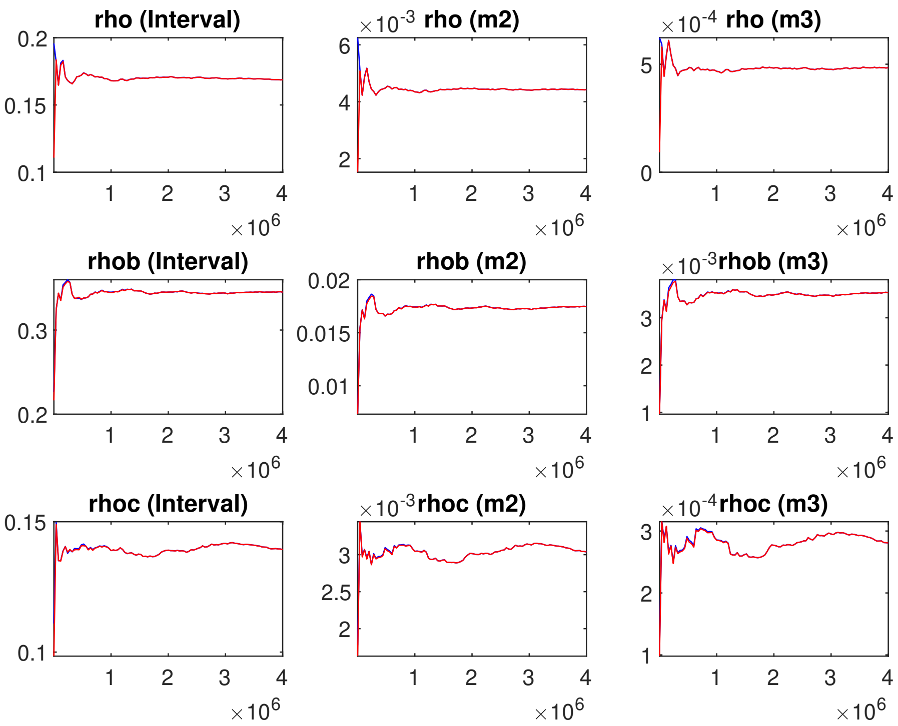

Figure 32: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

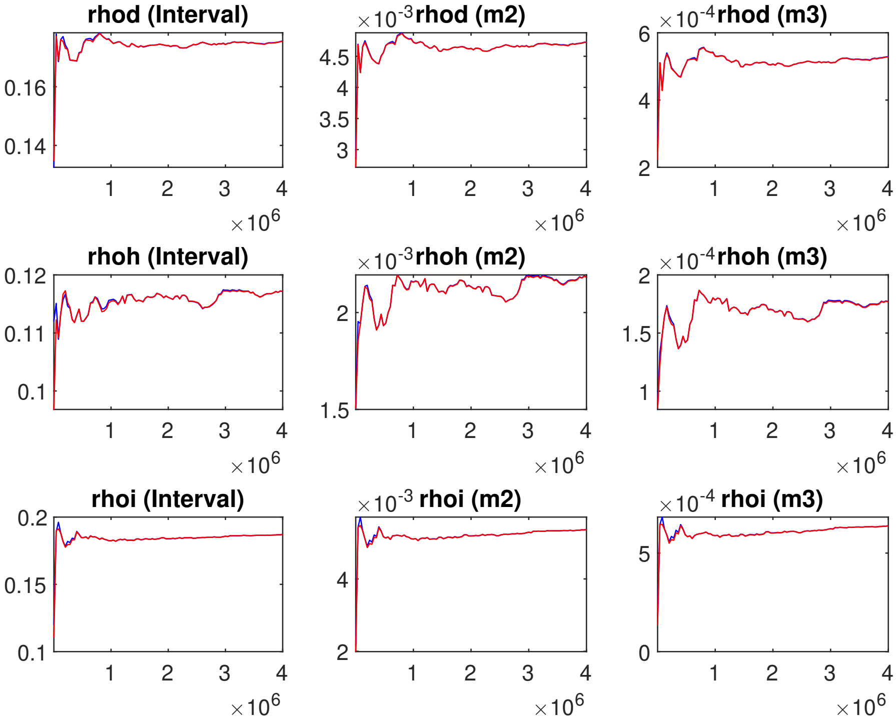

Figure 33: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

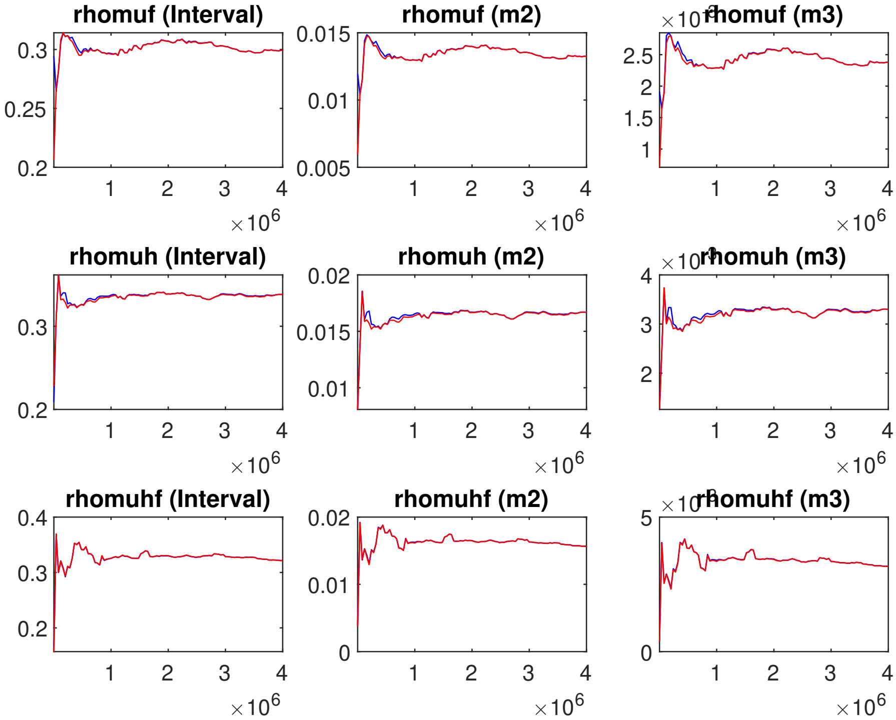

Figure 34: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

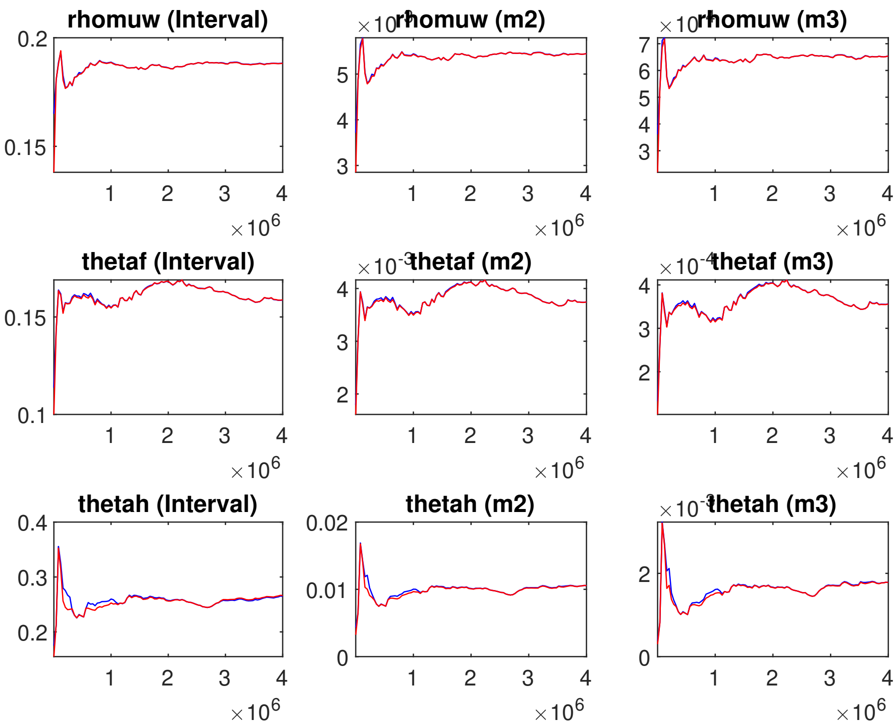

Figure 35: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

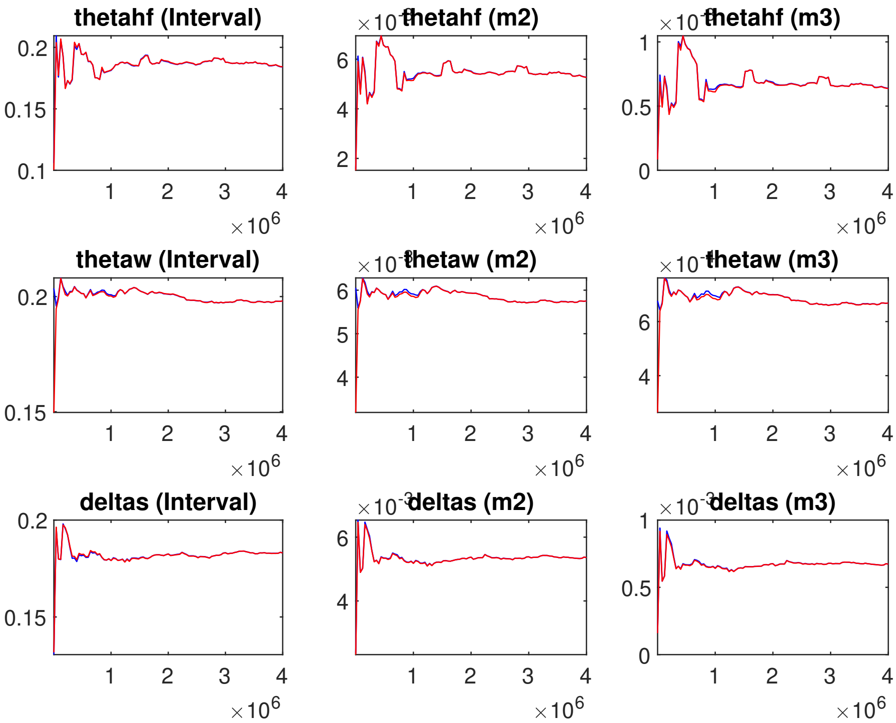

Figure 36: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

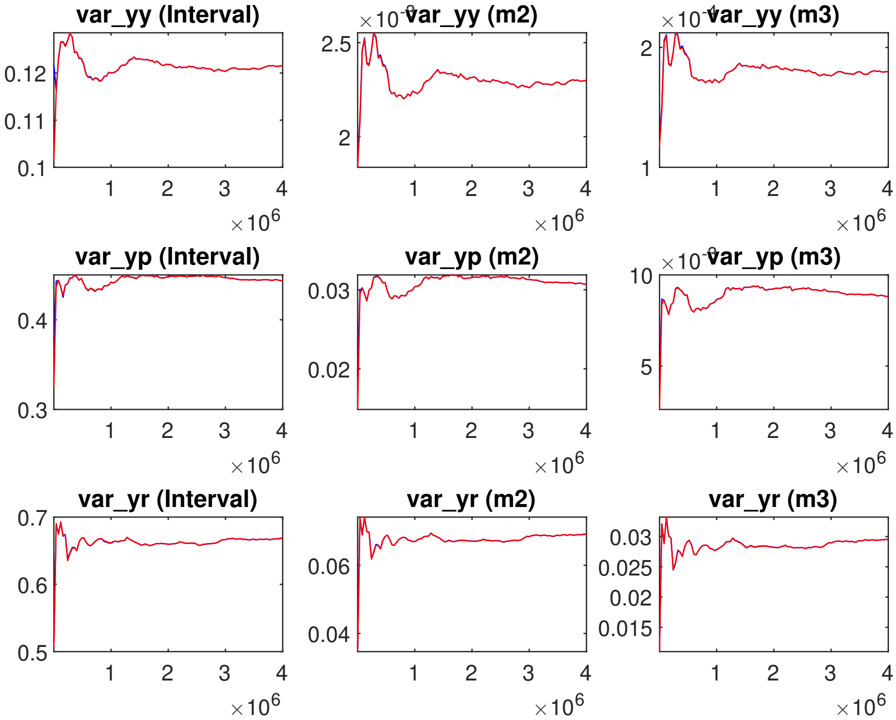

Figure 37: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

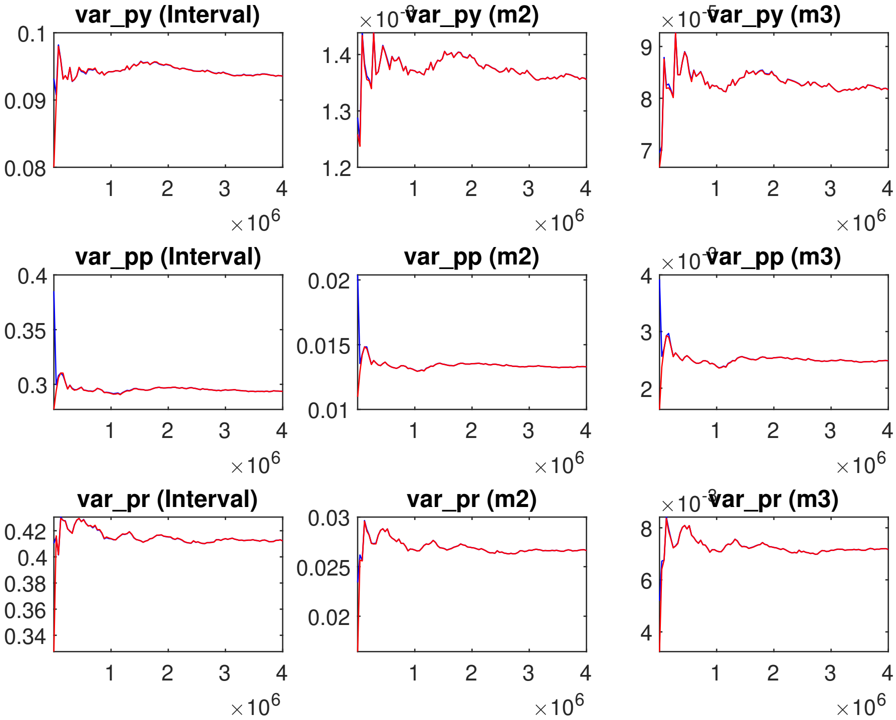

Figure 38: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

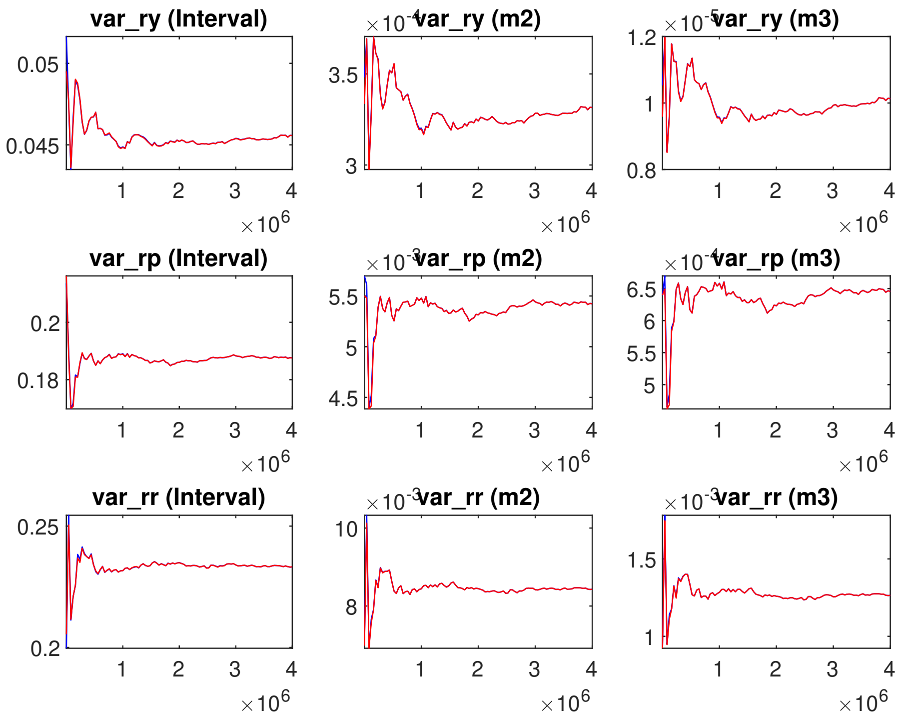

Figure 39: Univariate convergence diagnostics for the Metropolis-Hastings. The first, second and third columns are respectively the criteria based on the eighty percent interval, the second and third moments.

10.1.6 Smoothed observables

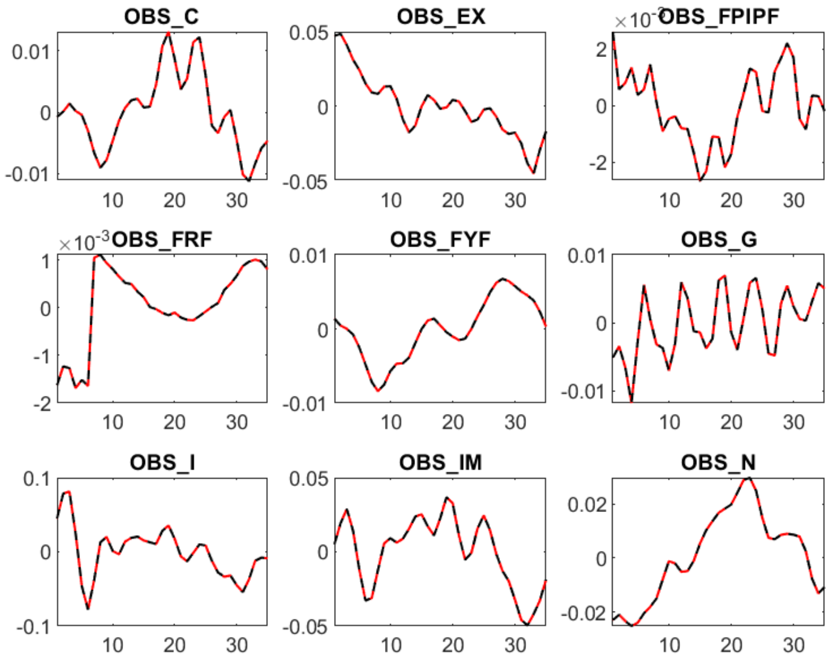

Figure 40: Smoothed observables.

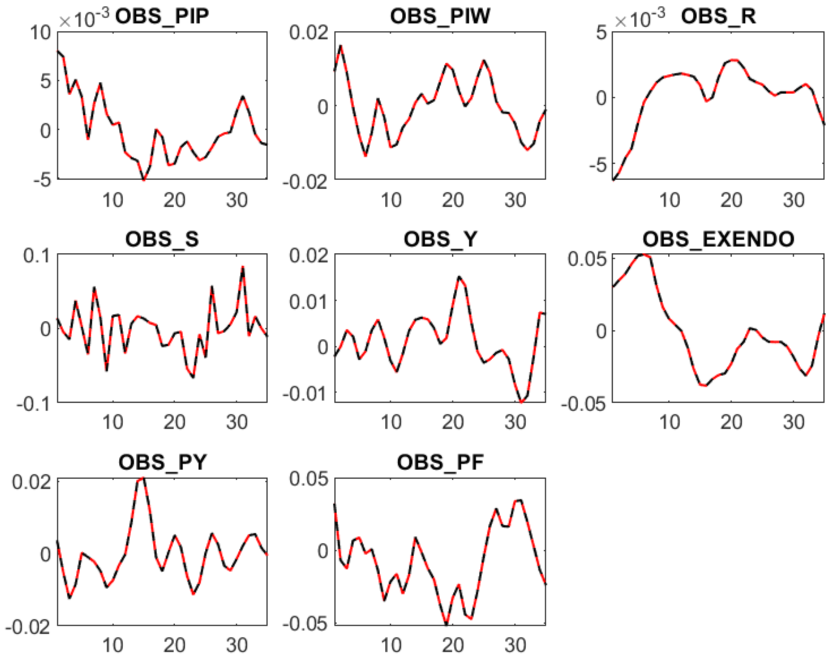

Figure 41: Smoothed observables.

10.2 Tables

Table 11: Priors and posteriors for parameters of the foreign economy's VAR model.

	Prior			Posterior			
	Dist.	Mean	Stdev.	Mean	Stdev.	HPD inf	HPD sup
$p_{y,y}$	norm	0.900	2.0000	0.956	0.0479	0.8785	1.0349
$p_{y,\pi}$	norm	0.000	2.0000	-0.193	0.1752	-0.4819	0.0933
$p_{y,r}$	norm	0.000	2.0000	0.322	0.2628	-0.1087	0.7501
$p_{\pi,y}$	norm	0.100	2.0000	0.048	0.0368	-0.0119	0.1087
$p_{\pi,\pi}$	norm	0.600	2.0000	0.614	0.1153	0.4256	0.8030
$p_{\pi,r}$	norm	-0.100	2.0000	-0.334	0.1632	-0.6015	-0.0678
$p_{r,y}$	norm	0.000	2.0000	-0.023	0.0182	-0.0530	0.0063
$p_{r,\pi}$	norm	0.000	2.0000	0.071	0.0736	-0.0510	0.1904
$p_{r,r}$	norm	0.800	2.0000	0.857	0.0918	0.7091	1.0070

Table 12: Correspondence between model variables and estimation variables.

Model variable	Estimation variable
σ_Z	SE_EPS
σ_G	SE_EPSG
σ_H	SE_EPSB
σ_C	SE_EPSC
σ_D	SE_EPSD
σ_I	SE_EPSI
σ_{μ_F}	SE_EPSMUF
σ_{μ_H}	SE_EPSMUH
$\sigma^*_{\mu_H}$	SE_EPSMUHF
σ_{μ_W}	SE_EPSMUW
σ_{P_F}	SE_EPSPF
σ_E	SE_EPSEX
σ^*_A	SE_EPSFYF
σ^*_π	SE_EPSFPIPF
σ^*_R	SE_EPSFRF
h_C	hc
h_N	hn
λ_I	lambdai
ϕ_B	phib
$\phi_{\Delta Y}$	phideltay

(Continued on next page)

Table 12: (continued)

Model variable	Estimation variable
ϕ_P	phip
ϕ_Y	phiy
θ_H	thetaf
θ_F	thetah
θ_F^*	thetahf
θ_W	thetaw
ξ	deltas
ρ_Z	rho
ρ_B	rhob
ρ_C	rhoc
ρ_D	rhod
ρ_H	rhoh
ρ_I	rhoi
ρ_{μ_F}	rhomuf
ρ_{μ_H}	rhomuh
$\rho_{\mu_H}^*$	rhomuhf
ρ_{μ_W}	rhomuw